

Birational morphisms and Poisson moduli spaces

Eric M. Rains

Department of Mathematics, California Institute of Technology

July 26, 2016

Abstract

We study birational morphisms between smooth projective surfaces that respect a given Poisson structure, with particular attention to induced birational maps between the (Poisson) moduli spaces of sheaves on those surfaces. In particular, to any birational morphism, we associate a corresponding “minimal lift” operation on sheaves of homological dimension ≤ 1 , and study its properties. In particular, we show that minimal lift induces a stratification of the moduli space of simple sheaves on the codomain by open subspaces of the moduli space of simple sheaves on the domain, compatibly with the induced Poisson structures.

Contents

1	Introduction	1
2	Poisson surfaces	3
3	Poisson moduli spaces	8
3.1	Locally free sheaves	11
3.2	Sheaves of homological dimension ≤ 1	14
4	The minimal lift	18
5	Invisible sheaves	22
6	Invisible sheaves and minimal lifts	30
7	Maps between Poisson moduli spaces	38
8	Rigidity	44

1 Introduction

If X is a smooth projective surface equipped with a Poisson structure, then the moduli space of sheaves on X inherits a corresponding Poisson structure ([4, 8], see below for precise details and a slight extension), and for any line bundle on X , twisting by that bundle preserves the Poisson structure on the moduli space. This fact underlies the construction of the (generalized) Hitchin integrable systems [9], which correspond to those sheaves on X which are the direct images of line bundles on smooth curves. If the curve meets the center of the Poisson structure (the curve where the Poisson structure fails to be symplectic) transversely, we can blow up a point of intersection,

and consider the image of the same line bundle on the strict transform. After finitely many such steps, one obtains a sheaf supported entirely on the open symplectic leaf of the surface, and the corresponding neighborhood of the moduli space is itself symplectic. In particular, one in this way obtains a large group acting as symplectomorphisms of a symplectic space, a discrete integrable system.

Recent developments [14] suggest that there should be an analogue of this picture for noncommutative surfaces, in which the integrable system is deformed to a generalized discrete Painlevé equation. The largest conceptual difficulty here is that direct images of line bundles are quite rare in the noncommutative picture, so we no longer have such a natural way to lift sheaves through a birational morphism. For that matter, even in the commutative setting, while being a direct image of a line bundle on a smooth curve is an open condition, the standard construction gives little insight into what happens outside this open subspace.

The objective of the present note is to give an alternate description for this lifting operation that immediately extends to arbitrary sheaves of homological dimension ≤ 1 . Although this does not quite give a morphism of moduli spaces (as it fails to preserve flatness of a family of sheaves), we do obtain a natural stratification of the “downstairs” moduli space such that the lifting operation is a morphism on each stratum, with open image. Moreover, in the Poisson case, the strata are all Poisson subspaces, and all morphisms are compatible with the Poisson structure.

The main ingredient in the construction is the observation that there are two other natural ways to lift a sheaf of homological dimension ≤ 1 through a birational morphism. The first is the usual inverse image: although this is larger than we want in the curve case, it is still a very well behaved functor. In particular, sheaves of homological dimension ≤ 1 on a surface are acyclic for the inverse image, and the direct image of the inverse image is naturally isomorphic to the original sheaf. The other natural lifting operation is slightly less obvious: it turns out that in the case of a birational morphism of smooth surfaces, the exceptional inverse image functor, normally only a functor on the derived categories, is itself a left-derived functor, and the resulting functor $\pi^!$ has the same nice properties as π^* . Adjointness then gives a natural transformation $\pi^* \rightarrow \pi^!$, and the lifting operation we want is the image of this natural transformation. (This can also be characterized as the minimal sheaf on the blown up surface which is acyclic and has the correct direct image.)

As part of our investigations of this “minimal lift” operation, we need to understand the “invisible” sheaves, those with trivial direct image and higher direct images, as these are precisely the sub- and quotient sheaves that have no effect on the direct image. It turns out that the invisible sheaves form an abelian subcategory of the category of (coherent) sheaves on the domain, equivalent to the category of (finitely generated) modules on a certain ring (a finite k -algebra). The kernel and cokernel of the natural map $\pi^*M \rightarrow \pi^!M$ are invisible, and a number of questions about the minimal lift reduce to questions about these sheaves. (Conversely, the equivalence of the category of invisible sheaves to a module category relies on a construction of projective objects as kernels of maps $\pi^*M \rightarrow \pi^!M$.)

The plan of the paper is as follows. First, we remind the reader of what a Poisson structure means in the algebraic setting (including in finite characteristic), and give a classification of (smooth, projective) Poisson surfaces up to birational equivalence respecting the Poisson structure. We find that every non-symplectic Poisson surface is birational to a Poisson ruled surface $\mathbb{P}(\mathcal{O}_C \oplus \omega_C)$ where C is a smooth curve, and the center of the Poisson structure is disjoint from the given section of the ruling. When $g(C) \geq 1$, these are precisely the surfaces that arise in generalized Hitchin systems; in the rational case, there are some additional cases not yet covered in the literature, which we will consider in more detail in [15]. Of course, the rational case is particularly interesting when it comes to birational maps, since only in that case do we obtain birational maps that do not preserve the ruling.

Next, we consider some technicalities involving the precise moduli space we will be using. Since in general the notion of stability requires a choice of ample bundle, it can be awkward to compare stability conditions on either side of a morphism; in addition, semistable points can interact badly with the Poisson structure. If we allow the moduli space to become slightly more complicated, we can relax “stable” to the more intrinsic “simple”, i.e., having no nonscalar endomorphisms. Although this space is no longer a scheme, we can still make sense of what it means for it to be a Poisson structure. The arguments in the literature mostly carry over to show that this is Poisson, and covered by smooth symplectic leaves; we use a reduction of [8] to the case of vector bundles, and give a new proof in that case that is somewhat simpler and extends with little difficulty to arbitrary characteristic.

We next consider how the two inverse image functors behave on sheaves of homological dimension ≤ 1 , allowing us to define the minimal lift of such a sheaf. We then investigate invisible sheaves, as indicated above. Following that, we prove our main result on the interaction between the minimal lift and the Poisson structure, and also investigate how certain natural “direct image of twist of minimal lift” operations interact with the Poisson structure. We conclude by considering as an application the question of rigidity: which sheaves on a Poisson surface are the unique point of their symplectic leaf? In the case of sheaves with 1-dimensional support, we show that the rigid sheaves are just the line bundles on -2 -curves of blowups. This is again mainly a feature of the rational surface case, and will be investigated further in [15].

Acknowledgements. The author would like to thank P. Etingof, T. Graber, and A. Okounkov for helpful conversations. This work was partially supported by a grant from the National Science Foundation, DMS-1001645.

2 Poisson surfaces

Recall that a Poisson algebra over a commutative ring R is a commutative R -algebra A equipped with an R -bilinear map $\{, \} : A \times A \rightarrow A$ which is a biderivation (a derivation in each variable) and satisfies the identities

$$\{f, f\} = 0, \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad (2.1)$$

for all elements f, g, h . (Note that we make no assumption about characteristic here.) By universality of Kähler differentials, to specify an alternating biderivation, it is equivalent to specify a homomorphism

$$\wedge^2 \Omega_{A/R} \rightarrow A. \quad (2.2)$$

of A -modules. Moreover, if $\{, \}$ is an alternating biderivation, then the left-hand side of the Jacobi identity is an alternating triderivation, which is required to vanish. A map $\pi : A \rightarrow B$ of Poisson R -algebras is called a Poisson map if

$$\pi(\{f, g\}) = \{\pi(f), \pi(g)\}; \quad (2.3)$$

we will also need to consider *anti-Poisson* maps, such that

$$\pi(\{f, g\}) = -\{\pi(f), \pi(g)\}. \quad (2.4)$$

These notions localize in a natural way, so one may thus define a Poisson structure on a scheme X/S to be a morphism

$$\alpha : \wedge^2 \Omega_{X/S} \rightarrow \mathcal{O}_X \quad (2.5)$$

of \mathcal{O}_X -modules such that the associated map

$$\wedge^3 \Omega_{X/S} \rightarrow \mathcal{O}_X \quad (2.6)$$

(defined locally in the obvious way) is 0. Note that this induces a Poisson algebra structure on $\Gamma(U, \mathcal{O}_X)$ for all open subsets U .

One can define Poisson and anti-Poisson morphisms of schemes accordingly. In scheme-theoretic terms, a Poisson morphism $(X, \alpha_X) \rightarrow (Y, \alpha_Y)$ is a morphism $f : X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} f^* \wedge^2 \Omega_{Y/S} & \xrightarrow{f^* \alpha_Y} & f^* \mathcal{O}_Y \\ \downarrow & & \parallel \\ \wedge^2 \Omega_{X/S} & \xrightarrow{\alpha_X} & \mathcal{O}_X \end{array} \quad (2.7)$$

commutes, and similarly, with appropriate sign changes, for anti-Poisson morphisms. Note that in general the identity morphism gives an anti-Poisson morphism $(X, \alpha) \cong (X, -\alpha)$. (We will also have occasion to use the obvious analogues for a general biderivation.)

Proposition 2.1. *Let X be a Poisson scheme, and suppose $f : X \rightarrow Y$ is a morphism such that the natural map $f_* \mathcal{O}_X \rightarrow \mathcal{O}_Y$ is an isomorphism. Then there is a unique Poisson structure on Y making f Poisson.*

Proof. The composition

$$f^* \wedge^2 \Omega_{Y/S} \rightarrow \wedge^2 \Omega_{X/S} \rightarrow \mathcal{O}_X \quad (2.8)$$

induces by adjointness a map

$$\wedge^2 \Omega_{Y/S} \rightarrow f_* \mathcal{O}_X \cong \mathcal{O}_Y, \quad (2.9)$$

i.e. an alternating biderivation. To check that this satisfies the Jacobi identity, we may reduce to the case $S = \text{Spec}(R)$, $Y = \text{Spec}(A)$ for an R -algebra A , in which case this biderivation is just the induced biderivation on $\Gamma(X, \mathcal{O}_X) \cong A$. But this is clearly a Poisson bracket. \square

Remark. Examples include a projective birational morphism $\pi : X \rightarrow Y$ with X integral and Y normal (and locally Noetherian), as well as the morphism $\rho : X \rightarrow C$ associated to a rationally ruled surface. (Of course, in the latter case, the induced Poisson structure is trivial!)

Proposition 2.2. *Let Y be a Poisson scheme, and suppose $f : X \rightarrow Y$ is a morphism such that $f^* \Omega_{Y/S} \rightarrow \Omega_{X/S}$ is an isomorphism (e.g. if f is étale). Then there is a unique Poisson structure on X making f Poisson.*

Proof. Clearly, the biderivation on X must be

$$\wedge^2 \Omega_{X/S} \rightarrow \wedge^2 \pi^* \Omega_{Y/S} \rightarrow \pi^* \mathcal{O}_Y \cong \mathcal{O}_X, \quad (2.10)$$

which is manifestly alternating. The isomorphism on differentials in particular implies that a derivation vanishes iff it vanishes on functions pulled back from Y (i.e., on pullbacks of sections of Y on open subsets U). In particular, it follows that the Jacobi triderivation vanishes as required. \square

A *Poisson subscheme* of a Poisson scheme $(X/S, \alpha)$ is a locally closed subscheme $Y/S \subset X/S$ which locally satisfies $\{f, \mathcal{I}_Y\} \subset \mathcal{I}_Y$ where \mathcal{I}_Y is the ideal sheaf of Y and f is any local section of \mathcal{O}_X . In particular, we immediately obtain an induced Poisson structure on Y such that the corresponding inclusion morphism is Poisson, and such a structure exists iff Y is a Poisson subscheme.

An important class of Poisson schemes are the symplectic schemes. A *symplectic scheme* is a pair (X, ω) such that X/S is smooth and $\omega \in \wedge^2 \Omega_{X/S}$ is a closed 2-form that induces an isomorphism $\Omega_{X/S} \cong T_{X/S}$. Call (X, ω) *presymplectic* if ω is nondegenerate, but not necessarily closed. Then ω induces a biderivation

$$\alpha : \Omega_{X/S} \otimes \Omega_{X/S} \xrightarrow{1 \otimes \omega} \Omega_{X/S} \otimes T_{X/S} \rightarrow \mathcal{O}_X, \quad (2.11)$$

which, as the inverse of an alternating form, is alternating. Conversely, if α is an alternating biderivation which is a nonsingular alternating form on $\Omega_{X/S}$ at every point, then it determines a presymplectic structure on X .

Proposition 2.3. *Suppose (X, ω) is a presymplectic scheme. Then (X, ω) is symplectic iff the corresponding biderivation is Poisson.*

Proof. Indeed, just as in characteristic 0, ω induces an isomorphism

$$\wedge^k \Omega_{X/S} \cong \wedge^k \text{Hom}(\Omega_{X/S}, \mathcal{O}_X) \cong \text{Hom}(\wedge^k \Omega_{X/S}, \mathcal{O}_X) \quad (2.12)$$

for any k , which for $k = 3$ takes $d\omega$ to the triderivation coming from the Jacobi identity. \square

Remark. In characteristic 0, it is well-known that a Poisson subscheme of a symplectic scheme is necessarily open. (An ideal of a symplectic local ring is Poisson iff it is preserved by all derivations, but this is impossible in characteristic 0 for a proper, nontrivial ideal.) This fails in finite characteristic; indeed, in characteristic p , the principal ideal generated by a p -th power will always be Poisson.

By a *Poisson surface*, we will mean a pair (X, α) where X/S is a projective surface with geometrically smooth and irreducible fibers, and α is a Poisson structure on X over S which is not identically 0 on any fiber. Note that since X is a surface,

$$\wedge^2 \Omega_{X/S} \cong \omega_{X/S} \quad (2.13)$$

and

$$\wedge^3 \Omega_{X/S} = 0, \quad (2.14)$$

and thus any section of the anticanonical bundle $\omega_{X/S}^{-1}$ determines a Poisson structure on X/S . Thus a nontrivial Poisson structure is determined up to a scalar by the divisor of the corresponding section of $\omega_{X/S}^{-1}$, the *associated anticanonical curve*, on the complement of which the surface is symplectic. The scalar can itself be identified: if C_α denotes the associated anticanonical curve, then the short exact sequence (which depends on α only via C_α)

$$0 \rightarrow \omega_X \rightarrow \omega_X \otimes \mathcal{L}(C_\alpha) \rightarrow \omega_{C_\alpha} \rightarrow 0 \quad (2.15)$$

induces an injection $H^0(\omega_X \otimes \mathcal{L}(C_\alpha)) \rightarrow H^0(\omega_{C_\alpha})$. A Poisson structure with associated anticanonical curve C_α is a nonzero global section of $\omega_X^{-1} \otimes \mathcal{L}(-C)$, so an isomorphism $\mathcal{O}_X \cong \omega_X^{-1} \otimes \mathcal{L}(-C)$. The inverse of this isomorphism determines a nonzero global section of $\omega_X \otimes \mathcal{L}(C_\alpha)$, and thus a nonzero holomorphic differential on C_α in the kernel of the connecting map $H^0(\omega_{C_\alpha}) \rightarrow H^1(\omega_X)$. Conversely, any nonzero element of that kernel determines a nonzero global section $\omega_X \otimes \mathcal{L}(C_\alpha)$, the inverse of which is a Poisson structure with associated anticanonical curve C_α . Note that the differential on C_α scales inversely with the Poisson structure.

As observed above, one can push Poisson structures forward through proper birational morphisms; in the case of Poisson surfaces, one can be fairly precise about lifting as well. Since everything is local on the base, we will assume $S = \text{Spec}(\bar{k})$ for some algebraically closed field \bar{k} . Note that if $\pi : X \rightarrow Y$ is a birational morphism of smooth projective surfaces over \bar{k} , then there is a (unique) effective divisor e_π supported on the exceptional locus such that $K_X \sim \pi^* K_Y + e_\pi$.

Lemma 2.4. *Let (Y, α) be a Poisson surface over \bar{k} , with associated anticanonical curve C , and suppose $\pi : X \rightarrow Y$ is a birational morphism. Then there is a (unique) Poisson structure on X compatible with that of Y iff the divisor $\pi^*C - e_\pi$ is effective.*

Proof. Indeed, $\pi^*C - e_\pi$ is the unique anticanonical divisor agreeing with C where π is an isomorphism, so if there is a Poisson structure on X , its associated anticanonical curve must be $\pi^*C - e_\pi$. In particular, $\pi^*C - e_\pi$ must be effective. If it is, then it determines a Poisson structure up to scalar multiplication, and that Poisson structure agrees with α (again up to scalar multiplication) where π is an isomorphism. We may thus eliminate the scalar freedom and obtain a Poisson structure agreeing with α at the generic point of X . \square

Proposition 2.5. *Let (X, α_X) , (Y, α_Y) be Poisson surfaces over \bar{k} , and suppose $f : X \rightarrow Y$ is a rational map which is Poisson at the generic points of X and Y . Then there exists a Poisson surface Z and birational Poisson morphisms $g : Z \rightarrow X$, $h : Z \rightarrow Y$ such that $f = h \circ g^{-1}$.*

Proof. Let $f = h \circ g^{-1}$ be the minimal factorization (i.e., Z is the minimal desingularization of the graph of f); we need to show that Z has a compatible Poisson structure. We can factor each of g and h as a product of monoidal transformations, and we can arrange that those monoidal transformations centered in points of the anticanonical curves come first. We thus reduce to the case that X and Y are isomorphic on some neighborhoods of their respective anticanonical curves, and wish to show that f is an isomorphism. Indeed, otherwise, we have an identity

$$g^*C_X - e_g = h^*C_Y - e_h \quad (2.16)$$

of divisors on Z . Since f is an isomorphism on neighborhoods of the anticanonical curves, $g^*C_X = h^*C_Y$, and thus $e_g = e_h$. If g is not an isomorphism, then e_g contains some -1 curve, which must be contracted by h , contradicting minimality of Z . \square

We in particular have a well-behaved notion of a Poisson birational map between Poisson surfaces. Since our objective in this note is to understand how moduli spaces of sheaves on Poisson surfaces are affected by Poisson birational maps, it will be convenient to give a classification of Poisson surfaces up to Poisson birational equivalence. (Compare [4, 3], which consider minimal Poisson surfaces, but without considering birational maps between them.) Call a Poisson surface *standard* if it is isomorphic as an algebraic surface to the ruled surface $\mathbb{P}(\mathcal{O}_C \oplus \omega_C)$ for some smooth curve C , in such a way that the morphism $\mathcal{O}_C \oplus \omega_C \rightarrow \omega_C$ determines a section disjoint from the anticanonical curve.

There is also an odd case in characteristic 2. Let C be a smooth genus 1 curve, and let X be the ruled surface corresponding to the nontrivial self-extension of \mathcal{O}_C . In characteristic different from 2, such a surface has a unique anticanonical curve, but in characteristic 2, there is actually an anticanonical pencil. Geometrically, we may take C to be in Weierstrass form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, and the vector bundle to have transition matrix

$$\begin{pmatrix} 1 & 0 \\ y/x & 1 \end{pmatrix} \quad (2.17)$$

between the complement of the identity and a suitable neighborhood of the identity. In general, a Poisson structure on $\mathbb{P}(V)$ corresponds to a global section of $S^2(V^*) \otimes \det(V) \otimes \omega_C^{-1}$, and thus in our case an anticanonical curve is cut out by a global section of $S^2(V^*)$. This has transition matrix

$$\begin{pmatrix} 1 & 0 & y^2/x^2 \\ 0 & 1 & y/x \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.18)$$

a nontrivial extension of \mathcal{O}_C^2 by \mathcal{O}_C . Since $\dim \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = 1$, this has precisely two dimensions of global sections. If z, w are the standard basis of the trivialization of V^* away from the identity, then the global sections of $S^2(V^*)$ are spanned by w^2 and $z^2 + a_1 zw + xw^2$. Indeed, if u, v are the basis of the other trivialization, then the latter becomes

$$u^2 + a_1 uv + (a_2 + a_3 y/x^2 + a_4/x + a_6/x^2)v^2, \quad (2.19)$$

the coefficients of which are holomorphic at the identity. Thus the generic anticanonical curve on this surface has the form $z^2 + a_1 z + c = x$ for some c . Note that we can replace c by $c + d^2 + a_1 d$ for any d , and thus all such anticanonical curves are related by geometric automorphisms of X . With that in mind, we call a Poisson surface of this form “quasi-standard”. Note that in general, the degree 2 map from C_α to C induces a 2-isogeny $J(C_\alpha) \rightarrow J(C)$ which is dual to the Frobenius 2-isogeny.

Theorem 2.6. *Any Poisson surface (X, α) over \bar{k} such that α is not an isomorphism is Poisson birational to a standard or quasi-standard Poisson surface.*

Proof. Since $-K_X$ is nontrivial and effective, X has Kodaira dimension $-\infty$, so either $X \cong \mathbb{P}^2$ or X admits a birational morphism to a ruled surface $\rho : X' \rightarrow C$. We may thus reduce to the case that X is a ruled surface; if $X \cong \mathbb{P}^2$, simply blow up a point of the anticanonical curve C_α . Since ρ admits infinitely many sections, we may choose a section in such a way that the corresponding curve is not a component of the anticanonical curve; by mild abuse of notation, we call this curve C .

If C intersects C_α , perform an elementary transformation (blow up a point, then blow down the fiber) based at a point of intersection. Blowing up the point reduces the intersection number by 1, and, since C meets the fiber only in the point being blown up, blowing down the fiber does not change the intersection number. Thus by induction, there is a sequence of Poisson elementary transformations making C disjoint from the anticanonical divisor, so we can reduce to the case that C was already disjoint from C_α .

In particular, $\omega_X \otimes \mathcal{O}_C \cong \mathcal{O}_C$, and thus we may use adjunction to compute $\omega_C \cong \mathcal{L}(C)|_C$. Applying ρ_* to the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{L}(C) \rightarrow \mathcal{L}(C)|_C \rightarrow 0 \quad (2.20)$$

gives the short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \rho_*(\mathcal{L}(C)) \rightarrow \omega_C \rightarrow 0, \quad (2.21)$$

where we observe that

$$X \cong \mathbb{P}(\rho_*(\mathcal{L}(C))). \quad (2.22)$$

We claim that this short exact sequence splits unless the surface is quasi-standard. If $g(C) = 0$, this is immediate. If $g(C) = 1$, then $-K_X \sim 2C$; since we also have $-K_X \sim C_\alpha$, we conclude that $h^0(-K_X) > 1$, implying that the sequence splits (or the surface is quasi-standard). Finally, if $g(C) \geq 2$, then the fact that $\chi(\mathcal{O}_{C_\alpha}) = 0$ and C_α has a degree 2 map to C (note that C_α cannot have a fiber as a component, since that would intersect C) implies that C_α cannot be integral. We can thus write $C_\alpha = C_1 + C_2$; since $C_\alpha \cdot f = 2$, where f is the (numerical) class of a fiber, we must have $C_1 \cdot f = C_2 \cdot f = 1$. But then C_1 and C_2 are sections disjoint from C , giving the desired splitting of $\rho_*(\mathcal{O}_X(C))$. (Such a splitting is moreover unique, so that $C_\alpha = 2C_1$.) \square

In particular, we have the following classification of Poisson surfaces, up to Poisson birational equivalence:

1. X is the Hirzebruch surface F_2 , and C_α is a reduced anticanonical curve disjoint from the minimal section of X .
2. X is $C \times \mathbb{P}^1$, with C a smooth genus 1 curve, and C_α is a union of two disjoint fibers over \mathbb{P}^1 .
3. X is $\mathbb{P}(\mathcal{O}_C \oplus \omega_C)$ for some smooth curve C , and $C_\alpha = 2C_0$, where C_0 is the section corresponding to the morphism $\mathcal{O}_C \oplus \omega_C \rightarrow \mathcal{O}_C$.
4. X is quasi-standard.
5. X is a minimal surface with trivial anticanonical bundle (K_3 , abelian, certain surfaces in characteristic 2 or 3 with nonreduced Pic^0 .)

Note that the reduced condition is only there to make cases 1 and 3 disjoint. Also, case 1 splits into subcases, corresponding to the Kodaira symbols I_0 , I_1 , II , I_2 , III (smooth, nodal integral, cuspidal integral, or two components meeting in a reduced or nonreduced scheme, respectively). Cases 2 and 3 and the reducible subcases of 1 have arisen in the theory of generalized Hitchin systems; for a related interpretation for rational surfaces with integral anticanonical curve, see [15].

Note that when the surface is not rational, either the anticanonical divisor is trivial and there are no nontrivial Poisson birational maps, or the (rational) ruling is uniquely determined, and thus any Poisson birational map between surfaces of the above canonical form is a composition of elementary transformations. In contrast, the rational case admits a rich structure of birational automorphisms respecting the Poisson structure, again see [15]. In particular, the I_1 and I_2 subcases are in the same birational equivalence class, as are the II and III subcases, though there are significant differences in the corresponding Hitchin-type systems.

We also note that the surfaces birational to quasi-standard Poisson surfaces can be characterized as those Poisson surfaces with rational rulings over genus 1 curves such that the anticanonical curve is integral.

3 Poisson moduli spaces

In [4, 5], Bottacin showed that the moduli space of stable vector bundles on a Poisson surface in characteristic 0 has a natural Poisson structure (extending a result of Tyurin [16], who constructed the form but did not prove the Jacobi identity), and identified the symplectic leaves of this structure (and in particular showed that they are algebraic). This was extended to general stable sheaves (in fact, simple sheaves), subject to some smoothness assumptions, in [8].

Since we wish to understand how these spaces interact with birational maps, we immediately encounter a problem: the definition of stability depends on a choice of ample line bundle. Since there is no canonical way to transport ample bundles through birational maps, this adds a great deal of complexity when trying to understand whether a given operation preserves stability. One way to avoid this is to replace “stable” with “simple”, having no nontrivial endomorphisms. As simplicity is intrinsic, we avoid any consideration of ample line bundles entirely, yet a result of Altman and Kleiman means that we still have a reasonably well-behaved moduli space.

Let X/S be locally projective, finitely presented morphism of schemes. A *family of simple sheaves* on X is a sheaf M on $T \times_S X$, flat over T , such that the fiber M over every geometric point $t \in T$ satisfies $\text{End}(M(t)) \cong k(t)$. We consider two such families M , M' equivalent if there is a line bundle \mathcal{L} on T such that $M' \cong M \otimes \pi_1^* \mathcal{L}$; this naturally preserves isomorphism classes of fibers. Note that the simplicity condition implies that the line bundle \mathcal{L} is unique (up to isomorphism) if

it exists, since for a simple family, we have

$$\mathrm{Hom}_{T \times_S X}(M, M \otimes \pi_1^* \mathcal{L}) \cong \Gamma(T \times_S X, \mathcal{L}) \quad (3.1)$$

Roughly speaking, this moduli problem is represented by an algebraic space. More precisely, this needs to be extended to the étale topology. Define a *twisted family of simple sheaves* on X parametrized by T to be a family of simple sheaves parametrized by some étale cover $U \rightarrow T$ such that the two induced families on $U \times_T U$ are equivalent.

Theorem 3.1. [1] *There is a quasi-separated algebraic space $\mathrm{Spl}_{X/S}$ locally finitely presented over S which represents the moduli functor of simple sheaves, in the sense that there is a natural bijection between twisted families of simple sheaves on X and morphisms to $\mathrm{Spl}_{X/S}$.*

In particular, we obtain such an algebraic space associated to any Poisson surface, and we wish to show that it is Poisson. Of course, this encounters another difficulty: what does it mean for an algebraic space to be Poisson? This is less of a difficulty than one might think, since as we have seen, Poisson structures can be pulled back through étale morphisms (and if a Poisson biderivation descends, the image is clearly Poisson). In other words, the notion of a Poisson structure makes perfect sense in the étale topology. We thus obtain the following definition of a Poisson structure on an algebraic space.

Definition 1. Let \mathcal{X} be an algebraic space. A Poisson structure on \mathcal{X} is an assignment of a Poisson structure to the domain of every étale morphism $f : U \rightarrow \mathcal{X}$, such that if $g : U' \rightarrow U$ is another étale morphism, then $g : (U', \alpha_{f \circ g}) \rightarrow (U, \alpha_f)$ is a Poisson morphism.

Remark. If \mathcal{X} is a Poisson scheme, then we obtain a Poisson structure in the above sense by assigning to every étale morphism the induced Poisson structure on its domain. Thus any Poisson scheme remains Poisson as an algebraic space.

Thus to obtain our Poisson moduli space, we need simply define a suitably canonical Poisson structure on the base of every twisted family of simple sheaves corresponding to an étale morphism to $\mathrm{Spl}_{X/S}$. In fact, it will be enough to consider only untwisted families, as the required compatibility of Poisson structures immediately gives us the conditions needed to descend the Poisson structure from the étale cover of the base of the twisted family. Note that a family corresponds to an étale morphism iff it is formally universal.

In [5], Bottacin showed that the moduli space of stable vector bundles on a Poisson surface has a well-behaved foliation by algebraic symplectic leaves: for any bundle V , the subscheme parametrizing stable vector bundles V' with $V'|_{C_\alpha} \cong V|_{C_\alpha}$ is a symplectic Poisson subscheme (where C_α is the anticanonical curve). This description does not quite carry over in full generality, although it is straightforward enough to fix: with this in mind, we define the *derived restriction* of a sheaf to be the complex

$$M|_{C_\alpha}^{\mathbf{L}} := \mathbf{L}i^* M, \quad (3.2)$$

where $i : C_\alpha \rightarrow X$ is the inclusion morphism.

We will mainly consider the case of a surface over a separably closed field, which we suppress from the notation.

Theorem 3.2. *Let (X, α) be a Poisson surface which is not symplectic and not birational to a quasi-standard Poisson surface. Then the moduli space Spl_X has a natural Poisson structure, and for any complex M_α^\bullet of locally free sheaves on the associated anticanonical curve C_α , the subspace of Spl_X parametrizing sheaves M with $M|_{C_\alpha}^{\mathbf{L}} \cong M_\alpha^\bullet$ is a (smooth) symplectic Poisson subspace.*

Remark 1. With this in mind, we define a *symplectic leaf* of $\mathcal{S}pl_X$ to be the subspace corresponding to some fixed derived restriction, or more generally a union of components of such a subspace.

Remark 2. Of course, this continues to hold when X has symplectic fibers (subject to the technical condition that $\text{Pic}^0(X)$ is smooth), with $\mathcal{S}pl_X$ itself smooth and symplectic. (See, e.g., Chapter 10 of [10].) The arguments below encounter difficulties in the symplectic case, however. The claim fails when X is birational to a quasi-standard surface.

Remark 3. Since the biderivation can be defined over an arbitrary base, and the further conditions to be a Poisson structure are closed on the base, we conclude that $\mathcal{S}pl_{X/S}$ is Poisson for any Poisson surface over a reduced scheme S , such that no fiber of X is symplectic. The claim for the symplectic leaves is harder to generalize, mainly since it is no longer true in general that conditions of the form $M|_{C_\alpha}^{\mathbf{L}} \cong M_\alpha$ are locally closed. It appears the correct condition to place on a family M_α over a reduced base is that it can be represented by a two-term perfect complex, and that

$$\dim \text{Hom}(M_\alpha, M_\alpha) - \dim \text{Ext}^{-1}(M_\alpha, M_\alpha) \quad (3.3)$$

is constant. (By the proof of Proposition 8.1 below, this condition is necessary for the tangent space to the symplectic leaf to have constant dimension.) With this proviso, the full Theorem most likely holds over an arbitrary reduced base, and a careful restatement of the constancy condition should extend to any locally Noetherian base. (For X rational (and probably for ruled surfaces over C with $g(C) = 1$), the reduced case is probably sufficient, since suitable moduli stacks of anticanonical surfaces are reduced; for $g(C) > 1$, though, most components of the moduli functor have obstructed deformations.)

In other words, we need to construct a Poisson structure on the base of every formally universal family of simple sheaves, and show that the foliation of the base by isomorphism class of derived restrictions to the anticanonical curve is a foliation by symplectic Poisson subspaces.

The first requirement is to compute the cotangent sheaf.

Lemma 3.3. *Let M be a formally universal family of simple sheaves on X parametrized by U . Then there is a natural isomorphism*

$$\Omega_U \cong \mathcal{E}xt_U^1(M, M \otimes \omega_X). \quad (3.4)$$

Proof. Fix an ample line bundle $\mathcal{O}_X(1)$ on X , and for each pair (m, n) of nonnegative integers, consider the fine moduli space of subsheaves $V \subset \mathcal{O}_X(-m)^{\oplus n}$ such that the quotient M is simple and acyclic for $R\text{Hom}(\mathcal{O}_X(-m), -)$, and the induced map

$$\text{Hom}(\mathcal{O}_X(-m), \mathcal{O}_X(-m)^{\oplus n}) \rightarrow \text{Hom}(\mathcal{O}_X(-m), M) \quad (3.5)$$

is an isomorphism. On the one hand, this moduli space can be directly identified as a disjoint union of open subschemes of Quot schemes, so Theorem 3.1 of [13] describes its cotangent sheaf. On the other hand, taking the quotient yields a map to $\mathcal{S}pl_X$ making it a principal PGL_n -bundle over an open subspace of $\mathcal{S}pl_X$. We may thus proceed as in the proof of Theorem 4.2 op. cit. to calculate the cotangent sheaf of that subspace of $\mathcal{S}pl_X$. Since as m, n varies these subspaces cover $\mathcal{S}pl_X$, the result follows. \square

The required biderivation on U is then given (following [16]) by

$$\Omega_U^{\otimes 2} \cong \mathcal{E}xt_U^1(M, M \otimes \omega_X)^{\otimes 2} \xrightarrow{1 \otimes \alpha} \mathcal{E}xt_U^1(M, M \otimes \omega_X) \otimes \mathcal{E}xt_U^1(M, M) \rightarrow \mathcal{O}_U, \quad (3.6)$$

where the last map is the trace pairing (i.e., Serre duality for the smooth morphism $X \times U \rightarrow U$). Note the commutative diagram

$$\begin{array}{ccc}
\mathcal{E}xt_U^1(M, M \otimes \omega_X)^{\otimes 2} & \xrightarrow{1 \otimes \alpha} & \mathcal{E}xt_U^1(M, M \otimes \omega_X) \otimes \mathcal{E}xt_U^1(M, M) \\
\downarrow \alpha \otimes 1 & & \downarrow \\
\mathcal{E}xt_U^1(M, M) \otimes \mathcal{E}xt_U^1(M, M \otimes \omega_X) & \longrightarrow & \mathcal{O}_U
\end{array} \tag{3.7}$$

Since the above biderivation is clearly functorial in U , and is invariant under twisting M by any line bundle, so in particular by line bundles pulled back from U , it remains only to show that it is Poisson, and to verify the claim about symplectic leaves.

Following [8], there are three cases to consider, depending on the homological dimension of the simple sheaf. Simple sheaves of homological dimension 2 are necessarily structure sheaves of closed points, and thus the corresponding component of $\mathcal{S}pl_X$ is naturally isomorphic to X itself, and the induced Poisson structure is the same. The case of simple sheaves of homological dimension < 2 can then be reduced to that of locally free simple sheaves. This reduction applies equally well in finite characteristic, though we will need to do some more work (beyond the arguments of [8]) to identify the symplectic leaves. We will consider this case shortly, but will first deal with locally free sheaves.

3.1 Locally free sheaves

The argument of [4] could most likely be carried over to finite characteristic, but is sufficiently complicated to make it somewhat difficult to verify this. It turns out, however, that Bottacin's identification of the symplectic leaves can be used to give a simpler proof which also easily carries over to finite characteristic.

We first need to know that the theorem applies to invertible sheaves.

Proposition 3.4. *The natural restriction map*

$$\mathrm{Pic}^0(X) \rightarrow \mathrm{Pic}^0(C_\alpha) \tag{3.8}$$

is an injective map of group schemes, and the natural Poisson structure is trivial on $\mathrm{Pic}(X) \subset \mathcal{S}pl_X$.

Proof. It follows from the previous classification that either X is rational, or there is a splitting of the induced map from C_α to the base C of the ruling. If X is rational, then $\mathrm{Pic}^0(X) = 1$, and there is nothing to prove, while if $C_\alpha \rightarrow C$ splits, then the composition

$$\mathrm{Pic}^0(X) \rightarrow \mathrm{Pic}^0(C_\alpha) \rightarrow \mathrm{Pic}^0(C) \tag{3.9}$$

agrees with the natural isomorphism.

Taking derivatives gives an injection

$$H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_{C_\alpha}) \tag{3.10}$$

implying that

$$H^1(\alpha) : H^1(\omega_X) \rightarrow H^1(\mathcal{O}_X) \tag{3.11}$$

is 0. But this implies for invertible M that the induced map

$$\mathrm{Ext}^1(M, M \otimes \omega_X) \rightarrow \mathrm{Ext}^1(M, M) \tag{3.12}$$

is 0, and thus so is the Poisson structure. \square

Remark. Note that this fails when X is birational to a quasi-standard Poisson surface. Indeed, in that case, the composition

$$\mathrm{Pic}^0(C) \cong \mathrm{Pic}^0(X) \rightarrow \mathrm{Pic}^0(C_\alpha) \quad (3.13)$$

is the Frobenius isogeny, and thus has nonreduced kernel (μ_2 or α_2 , depending on whether $J(C)$ is ordinary or supersingular). In particular, the fibers of restriction to C_α can fail to be smooth, and thus Theorem 3.2 fails in this case. Moreover, since the map $H^1(\omega_X) \rightarrow H^1(\mathcal{O}_X)$ induced by α is an isomorphism, we find that the natural biderivation on the 1-dimensional scheme $\mathrm{Pic}^0(X)$ induces a nondegenerate pairing on fibers of the cotangent bundle. In particular, the natural biderivation does not give a Poisson structure, as it is not even alternating. Note that we can exclude both this case and the symplectic cases on which the conclusion of Theorem 3.2 fails by insisting that the kernel of $\mathrm{Pic}^0(X) \rightarrow \mathrm{Pic}^0(C_\alpha)$ be reduced.

Corollary 3.5. *If X is rationally ruled over the smooth curve C , then $h^1(\mathcal{O}_{C_\alpha}) = g(C) + 1$.*

Proof. Indeed, one has a short exact sequence

$$0 \rightarrow H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_{C_\alpha}) \rightarrow H^2(\omega_X) \rightarrow 0 \quad (3.14)$$

where the first map is injective by the proposition. Since $h^1(\mathcal{O}_X) = g(C)$, $h^2(\omega_X) = 1$, the claim follows. \square

Remark. Again, in the quasi-standard case, this fails with $h^1(\mathcal{O}_{C_\alpha}) = 1$.

Now, let $\mathcal{V}ect_X$ denote the subspace of $\mathcal{S}pl_X$ classifying simple locally free sheaves; a sheaf being locally free is an open condition, so $\mathcal{V}ect_X$ is open in $\mathcal{S}pl_X$.

Lemma 3.6. *The algebraic space $\mathcal{V}ect_X$ is smooth.*

Proof. It suffices to prove that the moduli problem is formally smooth; i.e., any infinitesimal deformation of a vector bundle can be extended. But the obstructions to deformations of V are classified by

$$\mathrm{Ext}^2(V, V) \cong \mathrm{Hom}(V, V \otimes \omega_X)^* \cong \mathrm{Hom}(V, V \otimes \mathcal{L}(-C_\alpha))^*. \quad (3.15)$$

Since by assumption $\mathrm{Hom}(V, V)$ is generated by the identity, the injective map

$$\mathrm{Hom}(V, V \otimes \mathcal{L}(-C_\alpha)) \rightarrow \mathrm{Hom}(V, V) \quad (3.16)$$

must be 0. \square

In particular, since $\mathcal{V}ect_X$ is smooth, its tangent sheaf is well-behaved, and duality gives

$$\mathcal{T}_U \cong \mathcal{E}xt_U(V, V) \quad (3.17)$$

where V is any formally universal family with base U .

Lemma 3.7. *For any locally free sheaf V_α on C_α , the subspace of $\mathcal{V}ect_X$ on which $V \otimes \mathcal{O}_{C_\alpha} \cong V_\alpha$ is locally closed and smooth, and the corresponding conormal sheaf is the radical of the natural bilinear form on $\Omega_{\mathcal{V}ect_X}$.*

Proof. That the subspace is locally closed is standard. For smoothness, we need to show that any obstruction to a deformation inside this space must vanish. As in [4, 5], we obtain a self-dual long exact sequence the middle of which is

$$\rightarrow \mathrm{Hom}(V_\alpha, V_\alpha) \rightarrow \mathrm{Ext}^1(V, V \otimes \omega_X) \rightarrow \mathrm{Ext}^1(V, V) \rightarrow \mathrm{Ext}^1(V_\alpha, V_\alpha) \rightarrow \mathrm{Ext}^2(V, V \otimes \omega_X) \rightarrow (3.18)$$

Since \mathcal{Vect}_X is smooth, any infinitesimal deformation of V can be extended, and the question is whether the extension can be chosen in such a way that V_α remains constant. Equivalently, any such extension determines (as an extension of the trivial deformation of V_α) a class in $\mathrm{Ext}^1(V_\alpha, V_\alpha)$, which can be made trivial iff it is in the image of $\mathrm{Ext}^1(V, V)$, or equivalently iff its image in $\mathrm{Ext}^2(V, V \otimes \omega_X)$ is trivial. Since V is simple, the trace map

$$\mathrm{tr} : \mathrm{Ext}^2(V, V \otimes \omega_X) \rightarrow H^2(\omega_X) \quad (3.19)$$

is an isomorphism (it is dual to the natural map $k \rightarrow \mathrm{End}(V)$). It is thus equivalent to check whether the trace of the class in $\mathrm{Ext}^1(V_\alpha, V_\alpha)$ is in the image of $\mathrm{tr}(\mathrm{Ext}^1(V, V))$. But by [2], the trace map is the map induced by $V \mapsto \det(V)$ on the corresponding spaces of deformations. In particular the trace of the given class in $\mathrm{Ext}^1(V_\alpha, V_\alpha)$ is the class of the associated deformation of

$$\det(V_\alpha) \cong \det(V)|_{C_\alpha}. \quad (3.20)$$

We thus reduce to the case that V_α, V are invertible sheaves, in which case it follows by Proposition 3.4 that the deformation of V must be trivial, so certainly extends.

Now, by smoothness, U is an étale neighborhood in the given locally closed subspace iff we have an exact sequence

$$0 \rightarrow \mathcal{T}_U \rightarrow \mathcal{E}xt_U^1(V, V) \rightarrow \mathcal{E}xt_U^1(V_\alpha, V_\alpha), \quad (3.21)$$

where the first map is the Kodaira-Spencer map and the second map comes from the relative analogue of the above long exact sequence. Equivalently, the dual sequence

$$\mathcal{H}om_U(V_\alpha, V_\alpha) \rightarrow \mathcal{E}xt_U^1(V, V \otimes \omega_X) \rightarrow \Omega_U \rightarrow 0 \quad (3.22)$$

must be exact, and thus the conormal sheaf is equal to the radical, as required. \square

It follows that to check that the bracket is alternating and satisfies the Jacobi identity, it will suffice to verify this for the induced bracket on Ω_U with U an étale neighborhood in the subspace parametrizing locally free sheaves with $V|_{C_\alpha} \cong V_\alpha$. (Indeed, it then follows that the ideal sheaf measuring the failure to be Poisson is contained in the ideal sheaf of the symplectic leaf, and the intersection over all symplectic leaves is the trivial ideal sheaf.) Smoothness immediately implies that the induced bracket is alternating, as the self-pairing of a tangent vector is the obstruction to extending the corresponding deformation. Moreover, the claim about the radical implies that the corresponding map $\Omega_U \rightarrow \mathcal{T}_U$ is an isomorphism. In other words, the bracket induces a 2-form, and it remains only to show that this 2-form is closed.

Restricting to an open affine subscheme of U as convenient, we may assume that $V|_{C_\alpha}$ is isomorphic to the pullback of V_α (rather than merely isomorphic up to twisting by a line bundle on U). If we then choose an open affine covering U_i of X , we find that the corresponding transition functions g_{ij} are constant on C_α , and thus the natural Čech cocycle (representing the relevant portion of the Atiyah class of V , see [10] for a nice exposition)

$$\beta_{ij} := (d_U g_{ij}) g_{ij}^{-1} \in \mathcal{E}xt_U^1(V, V \otimes \Omega_U) \quad (3.23)$$

is trivial on C_α , so may be viewed as a cocycle in

$$\mathcal{E}xt_U^1(V, V \otimes \mathcal{L}(-C_\alpha) \otimes \Omega_U). \quad (3.24)$$

The 2-form associated to the above pairing is then given by the cocycle

$$\gamma_{ijk} := \text{Tr}(\beta_{ij} \wedge g_{ij} \beta_{jk} g_{ij}^{-1}) \in \mathcal{E}xt_U^2(V, V \otimes \mathcal{L}(-C_\alpha) \otimes \Omega_U^2) \cong \mathcal{E}xt_U^2(V, V \otimes \omega_X \otimes \Omega_U^2) \cong \Omega_U^2, \quad (3.25)$$

where the first isomorphism is induced by α , and the second is the trace pairing, and thus both isomorphisms are constant on U . Since

$$g_{ij} \beta_{jk} g_{ij}^{-1} = \beta_{ik} - \beta_{ij}, \quad (3.26)$$

$$d_U \beta_{ij} = -\beta_{ij} \wedge \beta_{ij}, \quad (3.27)$$

$$\text{Tr}(\beta_{ij} \wedge \beta_{ij}) = 0, \quad (3.28)$$

we can easily compute

$$d_U \gamma_{ijk} = d_U \text{Tr}(\beta_{ij} \wedge \beta_{ik}) \quad (3.29)$$

$$= -\text{Tr}(\beta_{ij} \wedge \beta_{ij} \wedge \beta_{ik}) - \text{Tr}(\beta_{ij} \wedge \beta_{ik} \wedge \beta_{ik}) \quad (3.30)$$

$$= -\check{d} \text{Tr}(\beta_{ij} \wedge \beta_{ij} \wedge \beta_{ij})/3, \quad (3.31)$$

and thus $d_U \gamma_{ijk}$ is actually a (Čech) coboundary of a cochain that vanishes to third order on C_α . It follows in particular that $d_U \gamma = 0$ in $\mathcal{E}xt_U^2(V, V \otimes \mathcal{L}(-C_\alpha)) \otimes \Omega_U^3$, as required.

This finishes our proof of the theorem for locally free sheaves.

Remark. Note that for a map $A : R^n \rightarrow V^n$ with R a commutative ring and V an R -module, we have

$$\text{Tr}(A \wedge A \wedge A) = \sum_{i,j,k} A_{ij} \wedge A_{jk} \wedge A_{ki} = 3 \sum_{i < j, k} A_{ij} \wedge A_{jk} \wedge A_{ki} \in \wedge^3(V). \quad (3.32)$$

This lets us define $\text{Tr}(A \wedge A \wedge A)/3$ even when 3 is not invertible, allowing the above argument to work in that case.

3.2 Sheaves of homological dimension ≤ 1

Now let $U \rightarrow \mathcal{S}pl_X$ be an étale neighborhood with corresponding family M of simple sheaves, and suppose that the fibers of M have homological dimension ≤ 1 . Let π denote the projection $U \times X \rightarrow U$.

Twisting M by a line bundle has no effect on the biderivation, so we may assume that the natural map $\pi^* \pi_* M \rightarrow M$ is surjective, and both M and $M \otimes \omega_X$ are π_* -acyclic. Then, as observed in [8], one obtains a corresponding locally free resolution

$$0 \rightarrow V \rightarrow W \rightarrow M \rightarrow 0 \quad (3.33)$$

with $W = \pi^* \pi_* M$ and V simple, such that we can reconstruct M from V . Indeed, acyclicity implies $\pi_* M$ is locally free, and the homological dimension condition then ensures that V is locally free. Moreover, the induced map

$$\mathcal{H}om_U(W, \mathcal{O}_X) \rightarrow \mathcal{H}om_U(V, \mathcal{O}_X) \quad (3.34)$$

is an isomorphism, and thus given V we can recover M as the cokernel of the natural injection

$$V \rightarrow \mathcal{H}om_U(\mathcal{H}om_U(V, \mathcal{O}_X), \mathcal{O}_X). \quad (3.35)$$

It follows in particular that V is also a family of simple sheaves, giving an injective morphism $U \rightarrow \mathcal{V}ect_X$, and we need to compare the two induced biderivations, then understand the symplectic leaves.

For the first part, we use a mild variant of the calculation in [8]. The functoriality of long exact sequences gives a commutative diagram

$$\begin{array}{ccc} \mathcal{H}om_U(V, M) & \longrightarrow & \mathcal{E}xt_U^1(M, M) \\ \downarrow & & \downarrow \\ \mathcal{E}xt_U^1(V, V) & \longrightarrow & \mathcal{E}xt_U^2(M, V) \end{array} \quad (3.36)$$

Here, the top arrow is surjective, since

$$\mathcal{E}xt_U^1(W, M) \cong \mathcal{H}om(\pi_*(M), R^1\pi_*(M)) = 0, \quad (3.37)$$

and the right arrow is injective since

$$\mathcal{E}xt_U^1(M, W) \cong \mathcal{H}om(R^1\pi_*(M \otimes \omega_X), \pi_*M) = 0. \quad (3.38)$$

There is also a natural map $\delta : \mathcal{E}xt_U^1(M, M) \rightarrow \mathcal{E}xt_U^1(V, V)$ coming from the above construction (the derivative of the map of moduli spaces). Since the top arrow above is surjective, we may represent any self-extension of M as the pushforward of the resolution $0 \rightarrow V \rightarrow W \rightarrow M \rightarrow 0$ by some morphism $\phi : V \rightarrow M$. This gives a self-extension M' as the cokernel in the short exact sequence

$$0 \rightarrow V \rightarrow W \oplus M \rightarrow M' \rightarrow 0. \quad (3.39)$$

The surjection $W \rightarrow M$ then induces a surjection $W \oplus W \rightarrow M'$, so an exact sequence

$$0 \rightarrow V' \rightarrow W \oplus W \rightarrow M' \rightarrow 0, \quad (3.40)$$

where V' is a self-extension of V (the image of M' under δ). Pulling this back through the injection $M \rightarrow M'$ gives a short exact sequence

$$0 \rightarrow V' \rightarrow V \oplus W \rightarrow M \rightarrow 0, \quad (3.41)$$

so that V' is the pullback through ϕ of $0 \rightarrow V \rightarrow W \rightarrow M \rightarrow 0$.

In other words, given a class in $\mathcal{E}xt_U^1(M, M)$, we can compute its image under δ by choosing a preimage in $\mathcal{H}om_U(V, M)$ and mapping that to $\mathcal{E}xt_U^1(V, V)$. In particular, we can put the differential into the diagram without breaking commutativity. Note in particular that commutativity implies that the differential is injective, and in fact identifies $\mathcal{E}xt_U^1(M, M)$ with the image of the map $\mathcal{H}om_U(V, M) \rightarrow \mathcal{E}xt_U^1(V, V)$. We similarly have a commutative square

$$\begin{array}{ccc} \mathcal{H}om_U(V, M \otimes \omega_X) & \longrightarrow & \mathcal{E}xt_U^1(M, M \otimes \omega_X) \\ \downarrow & & \downarrow \\ \mathcal{E}xt_U^1(V, V \otimes \omega_X) & \longrightarrow & \mathcal{E}xt_U^2(M, V \otimes \omega_X) \end{array} \quad (3.42)$$

Since the connecting maps in this diagram are the duals (up to sign) of the maps in the original diagram, we see that the dual of the differential also fits into this diagram; more precisely, we must take the negative of the dual in order to account for the signs.

Without the diagonal maps, the two squares fit into a commutative cube induced by $\alpha : \omega_X \rightarrow \mathcal{O}_X$. We claim that this diagram remains commutative when we introduce the diagonal maps. In other words, we need to show that the composition

$$\mathcal{E}xt_U^1(V, V \otimes \omega_X) \xrightarrow{-\delta^*} \mathcal{E}xt_U^1(M, M \otimes \omega_X) \xrightarrow{\alpha} \mathcal{E}xt_U^1(M, M) \xrightarrow{\delta} \mathcal{E}xt_U^1(V, V) \quad (3.43)$$

agrees with the map induced by α . If we compose both maps with the connecting map $\mathcal{E}xt_U^1(V, V) \rightarrow \mathcal{E}xt_U^2(M, V)$, they agree. If we can show that the image of $\mathcal{E}xt_U^1(V, V \otimes \omega_X) \rightarrow \mathcal{E}xt_U^1(V, V)$ is contained in the image of δ , we will be done, since then the error is in the image of δ , which injects in $\mathcal{E}xt_U^2(M, V)$. The image of δ is the same as the image of $\text{Hom}_U(V, M)$, and thus is the same as the kernel of the map $\mathcal{E}xt_U^1(V, V) \rightarrow \mathcal{E}xt_U^1(V, W)$. By commutativity of long exact sequences, it will thus suffice to show that the map $\mathcal{E}xt_U^1(V, W \otimes \omega_X) \rightarrow \mathcal{E}xt_U^1(V, W)$ is 0. We have a commutative square

$$\begin{array}{ccc} \mathcal{E}xt_U^1(W, W \otimes \omega_X) & \xrightarrow{\alpha} & \mathcal{E}xt_U^1(W, W) \\ \downarrow & & \downarrow \\ \mathcal{E}xt_U^1(V, W \otimes \omega_X) & \xrightarrow{\alpha} & \mathcal{E}xt_U^1(V, W) \end{array} \quad (3.44)$$

The left arrow is an isomorphism, by duality from the fact that $R^1\pi_*(V) \cong R^1\pi_*(W)$, while the top arrow is 0 by Proposition 3.4, since $\pi_*\omega_X \rightarrow \pi_*\mathcal{O}_X$ is the 0 morphism. It follows that the bottom arrow is 0 as required.

We have thus shown that the above map of moduli spaces identifies the two biderivations up to sign, and thus since the biderivation on $\mathcal{V}ect_X$ is Poisson, so is the biderivation on U . (The minus sign above makes this an anti-Poisson morphism.) There are two things remaining for us to do: first, we need to show that any locally closed subscheme of U on which $M|_{C_\alpha}^{\mathbf{L}}$ is fixed is a Poisson subscheme, and second that it is symplectic. Of course, we will show both at once if we can identify the putative leaves of U with the leaves of $\mathcal{V}ect_X$.

Now, $M|_{C_\alpha}^{\mathbf{L}}$ is represented by the complex $V_\alpha \rightarrow W_\alpha$, where V_α and W_α are the respective restrictions to C_α . We claim that, just as V determines M , V_α determines $M|_{C_\alpha}^{\mathbf{L}}$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(W, \mathcal{O}_X) & \longrightarrow & \text{Hom}(W_\alpha, \mathcal{O}_{C_\alpha}) & \longrightarrow & \text{Ext}^1(W, \omega_X) \longrightarrow \text{Ext}^1(W, \mathcal{O}_X) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Hom}(V, \mathcal{O}_X) & \longrightarrow & \text{Hom}(V_\alpha, \mathcal{O}_{C_\alpha}) & \longrightarrow & \text{Ext}^1(V, \omega_X) \longrightarrow \text{Ext}^1(V, \mathcal{O}_X) \end{array} \quad (3.45)$$

coming from the exact sequence

$$0 \rightarrow \omega_X \xrightarrow{\alpha} \mathcal{O}_X \rightarrow \mathcal{O}_{C_\alpha} \rightarrow 0. \quad (3.46)$$

Each vertical map fits into a long exact sequence; since M and $M \otimes \omega_X$ are acyclic, we may apply Serre duality to find that the first and third vertical maps are isomorphisms, while the fourth map is injective. We thus find that the given map $V_\alpha \rightarrow W_\alpha$ induces an isomorphism

$$\text{Hom}(W_\alpha, \mathcal{O}_{C_\alpha}) \cong \text{Hom}(V_\alpha, \mathcal{O}_{C_\alpha}). \quad (3.47)$$

In particular, $\text{Hom}(V_\alpha, \mathcal{O}_{C_\alpha})$ is a free $\text{End}(\mathcal{O}_{C_\alpha})$ -module, and the map $V_\alpha \rightarrow W_\alpha$ can be identified with the natural map

$$V_\alpha \rightarrow \text{Hom}_{\text{End}(\mathcal{O}_{C_\alpha})}(\text{Hom}(V_\alpha, \mathcal{O}_{C_\alpha}), \mathcal{O}_{C_\alpha}). \quad (3.48)$$

It follows that the symplectic leaves of \mathcal{Vect}_X pull back to closed subspaces of the putative symplectic leaves of U .

Conversely, we can recover V_α from $M|_{C_\alpha}^{\mathbf{L}}$ and the numerical invariants of M . (This also implies that the condition $M|_{C_\alpha}^{\mathbf{L}} \cong M_\alpha$ is locally closed.) Since M and $M \otimes \omega_X$ are acyclic, we have a four-term exact sequence

$$0 \rightarrow H^{-1}(M|_{C_\alpha}^{\mathbf{L}}) \rightarrow H^0(M \otimes \omega_X) \rightarrow H^0(M) \rightarrow H^0(M|_{C_\alpha}^{\mathbf{L}}) \rightarrow 0, \quad (3.49)$$

and thus (locally on the base) we may write

$$W_\alpha \cong H^0(M|_{C_\alpha}^{\mathbf{L}}) \otimes \mathcal{O}_{C_\alpha} \oplus \mathcal{O}_{C_\alpha}^l \quad (3.50)$$

where

$$l = h^0(M|_{C_\alpha}^{\mathbf{L}}) - h^0(M) = h^0(M|_{C_\alpha}^{\mathbf{L}}) - \chi(M). \quad (3.51)$$

Moreover, this isomorphism identifies the map $W_\alpha \rightarrow M|_{C_\alpha}^{\mathbf{L}}$ with the direct sum of the natural morphism

$$H^0(M|_{C_\alpha}^{\mathbf{L}}) \otimes \mathcal{O}_{C_\alpha} \rightarrow M|_{C_\alpha}^{\mathbf{L}} \quad (3.52)$$

and the zero morphism

$$\mathcal{O}_{C_\alpha}^l \rightarrow M|_{C_\alpha}^{\mathbf{L}}. \quad (3.53)$$

But then we can recover V_α as the “kernel” of this morphism; more precisely, the mapping cone of this morphism is a complex representing the sheaf V_α . In particular, we have

$$V_\alpha \cong \mathcal{O}_{C_\alpha}^l \oplus V_{\min}, \quad (3.54)$$

where

$$V_{\min} \rightarrow H^0(M|_{C_\alpha}^{\mathbf{L}}) \otimes \mathcal{O}_{C_\alpha} \quad (3.55)$$

is the canonical complex quasi-isomorphic to $M|_{C_\alpha}^{\mathbf{L}}$. (Of course, when $\mathcal{T}or_1(M, \mathcal{O}_{C_\alpha}) = 0$, so $M|_{C_\alpha}^{\mathbf{L}}$ is a sheaf, V_{\min} is an actual kernel, of the natural (surjective!) morphism $H^0(M|_{C_\alpha}^{\mathbf{L}}) \otimes \mathcal{O}_{C_\alpha} \rightarrow M|_{C_\alpha}^{\mathbf{L}}$.)

We thus conclude that the putative symplectic leaf of \mathcal{Spl}_X containing M can be identified with a Poisson subspace of a symplectic leaf of \mathcal{Vect} . It remains only to show that this Poisson subspace is open. (This is automatic in characteristic 0, since then any Poisson subspace of a symplectic space is open. The reader only interested in characteristic 0 may still find the direct argument instructive, however.)

Fix a simple sheaf M satisfying the above conditions, with corresponding locally free sheaf V . In the symplectic leaf of \mathcal{Vect}_X corresponding to V_α , we first impose the following open conditions on the fibers V' :

1. V' has the same numerical Chern class as V . (This is both open and closed.)
2. $H^0(V') = H^2(V') = 0$. (This is open by semicontinuity)
3. $\text{Ext}^2(V', \mathcal{O}_X) = 0$ and $\dim \text{Hom}(V', \mathcal{O}_X) \leq \text{rank}(W)$. (Also semicontinuity).

If U' is an étale neighborhood in this open subspace, then $\mathcal{E}xt_{U'}^1(V', \omega_X)$ is locally free, since the other Ext groups are trivial, and thus it has the same rank as

$$\text{Ext}^1(V, \omega_X) \cong \text{Ext}^1(W, \omega_X) \cong H^1(W)^* \quad (3.56)$$

Since W is trivial, we conclude

$$\text{rank}(\mathcal{E}xt_{U'}^1(V', \omega_X)) = g \text{rank}(W), \quad (3.57)$$

where $g = \dim H^1(\mathcal{O}_X)$ is the genus of the curve over which X is rationally ruled. Also, since V_α, W_α are fixed and $h^0(\mathcal{O}_{C_\alpha}) = g + 1$, we may compute

$$h^0(V_\alpha^*) = h^0(W_\alpha^*) = (g + 1) \operatorname{rank}(W). \quad (3.58)$$

But then the long exact sequence

$$0 \rightarrow \mathcal{H}om_{U'}(V', \mathcal{O}_X) \rightarrow \mathcal{H}om_{U'}(V', \mathcal{O}_{C_\alpha}) \rightarrow \mathcal{E}xt_{U'}^1(V', \omega_X) \rightarrow \cdots \quad (3.59)$$

implies that $\mathcal{H}om_{U'}(V', \mathcal{O}_X)$ contains a locally free sheaf of rank at least $\operatorname{rank}(W)$. Since we have already imposed an upper bound on the rank, we conclude that $\mathcal{H}om_{U'}(V', \mathcal{O}_X)$ is locally free of rank $\operatorname{rank}(W)$. We thus recover an injective map

$$V' \rightarrow \mathcal{H}om_{U'}(\mathcal{H}om_{U'}(V', \mathcal{O}_X), \mathcal{O}_X). \quad (3.60)$$

If we let M' be the cokernel of this map, then it certainly is generated by global sections, and we can further impose the open conditions that M' and $M' \otimes \omega_X$ are π_* -acyclic. We thus obtain a neighborhood of V in its symplectic leaf such that every sheaf in that neighborhood comes from a sheaf M' .

This completes the proof of Theorem 3.2. \square

4 The minimal lift

Much of what we have to say about the minimal lift construction is independent of the Poisson structure, so we will for the moment drop the assumption that our surfaces are Poisson. They also involve only local considerations, so we drop the requirement that the surfaces be projective, but also restrict to geometric fibers. Thus for the purposes of Sections 4-6, an “algebraic surface” will mean an irreducible smooth 2-dimensional scheme over an algebraically closed field. In addition, when we say that $\pi : X \rightarrow Y$ is a birational morphism, we will mean that X and Y are algebraic surfaces and π is projective.

Given a birational morphism $\pi : X \rightarrow Y$, there is an obvious way to transport coherent sheaves on Y to coherent sheaves on X , namely the inverse image functor π^* . This is very well-behaved on sheaves of homological dimension ≤ 1 .

Lemma 4.1. *Let $\pi : X \rightarrow Y$ be a birational morphism. Then any coherent sheaf M on Y of homological dimension ≤ 1 is π^* -acyclic. Moreover, the sheaf π^*M is π_* -acyclic, and the natural map $M \rightarrow \pi_*\pi^*M$ is an isomorphism.*

Proof. If π is monoidal and $M = \mathcal{O}_Y$, this is a standard fact about birational morphisms of surfaces. The result for general π follows by induction along a factorization of π into monoidal transformations, and the case M locally free follows easily.

In general, choose a locally free resolution

$$0 \rightarrow V \xrightarrow{B} W \rightarrow M \rightarrow 0. \quad (4.1)$$

Applying π^* gives an exact sequence

$$\pi^*V \xrightarrow{\pi^*B} \pi^*W \rightarrow \pi^*M \rightarrow 0, \quad (4.2)$$

with π^*V and π^*W locally free. Since B is injective, π^*B is still injective on the generic fiber; since π^*V is locally free, we conclude that π^*B is injective, and thus M is π^* -acyclic as required. The remaining claim follows upon taking the direct image of the resulting short exact sequence. \square

Regarding sheaves of homological dimension ≤ 1 , we have the following characterization.

Proposition 4.2. *A coherent sheaf M on an algebraic surface X has homological dimension ≤ 1 iff it has no subsheaf of the form \mathcal{O}_p for p a closed point of X .*

Proof. We note that M has homological dimension ≤ 1 iff $\mathcal{T}or_2(M, \mathcal{O}_{\bar{x}}) = 0$ for all (not necessarily closed) points $x \in X$. If $\dim(\bar{x}) = 2$, then $\mathcal{O}_{\bar{x}} = \mathcal{O}_X$, so there is no condition, while if $\dim(\bar{x}) = 1$, then \bar{x} is a divisor on X , and its structure sheaf has homological dimension 1. In other words, we find that M has homological dimension ≤ 1 iff $\mathcal{T}or_2(M, \mathcal{O}_p) = 0$ for all closed points p . Now, the minimal resolution of \mathcal{O}_p in the local ring at p is self-dual, and thus we have an isomorphism

$$\mathcal{T}or_2(M, \mathcal{O}_p) \cong \mathcal{H}om_X(\mathcal{O}_p, M). \quad (4.3)$$

Since $\mathcal{H}om_X(\mathcal{O}_p, M)$ is supported on p , $\mathcal{H}om_X(\mathcal{O}_p, M) = 0$ iff $\mathcal{H}om(\mathcal{O}_p, M) = 0$, iff \mathcal{O}_p is not a subsheaf of M . \square

Since $\pi_*\pi^*M = 0$, if π_* had a right adjoint $\pi^!$, then the isomorphism $\pi_*\pi^*M \cong M$ would induce morphisms $\pi^*M \rightarrow \pi^!M$ and $M \rightarrow \pi_*\pi^!M$. Of course, since π_* is not right exact, it cannot have a right adjoint, but duality theory gives an adjoint to the derived functor. In our case, we actually obtain something stronger.

Lemma 4.3. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces. Then there exists a right exact functor $\pi^! : \text{Coh}_Y \rightarrow \text{Coh}_X$, acyclic on locally free sheaves, such that for any coherent sheaves M, N on X and Y respectively (or bounded complexes of such sheaves),*

$$\mathbf{R}\mathcal{H}om(\mathbf{R}\pi_*M, N) \cong \mathbf{R}\mathcal{H}om_Y(M, \mathbf{L}\pi^!N). \quad (4.4)$$

Moreover, there is a natural isomorphism

$$N \cong \mathbf{R}\pi_*\mathbf{L}\pi^!N. \quad (4.5)$$

Proof. Since π_* is proper, $\pi_* = \pi_!$, and thus $\mathbf{R}\pi_*$ has a right adjoint $\pi^!$ on the derived category. Since X and Y are smooth,

$$\omega_X \cong (\phi \circ \pi)^!\mathcal{O}_{\bar{k}}[-2] \cong \pi^!(\phi^!\mathcal{O}_{\bar{k}}[-2]) \cong \pi^!\omega_Y, \quad (4.6)$$

where $\phi : Y \rightarrow \text{Spec}(\mathcal{O}_{\bar{k}})$ is the structure morphism of the \bar{k} -scheme Y . But then for any locally free sheaf V on Y ,

$$\pi^!V \cong \pi^!(\omega_Y \otimes (\omega_Y^{-1} \otimes V)) \cong \pi^!\omega_Y \otimes \mathbf{L}\pi^*(\omega_Y^{-1} \otimes V) \cong \pi^*V \otimes \omega_X \otimes \pi^*\omega_Y^{-1} \quad (4.7)$$

and thus in particular $\pi^!V$ is a sheaf. Using a locally free resolution, we conclude that

$$\pi^!N^\cdot \cong \mathbf{L}\pi^*N^\cdot \otimes \omega_X \otimes \pi^*\omega_Y^{-1} \quad (4.8)$$

for any complex N^\cdot . In other words, the derived functor $\pi^!$ is the left derived functor of the right exact functor

$$N \mapsto \pi^*N \otimes \omega_X \otimes \pi^*\omega_Y^{-1}. \quad (4.9)$$

For the remaining claim, we have

$$\mathbf{R}\pi_*\mathbf{L}\pi^!N \cong \mathbf{R}\mathcal{H}om_Y(\mathcal{O}_X, \mathbf{L}\pi^!N) \cong \mathbf{R}\mathcal{H}om(\mathbf{R}\pi_*\mathcal{O}_X, N) \cong \mathbf{R}\mathcal{H}om(\mathcal{O}_Y, N) \cong N. \quad (4.10)$$

\square

Remark. More generally, if $\pi : X \rightarrow Y$ is a proper morphism of Gorenstein varieties, then

$$\pi^! N \cong \mathbf{L}\pi^* N \otimes \omega_X \otimes \pi^* \omega_Y^{-1} [\dim(X) - \dim(Y)]; \quad (4.11)$$

this is essentially an observation of Deligne [6, Prop. 7]. Similarly, acyclicity on sheaves of homological dimension 1 and the isomorphisms $\mathbf{R}\pi_* \mathbf{L}\pi^* N \cong N$, $\mathbf{R}\pi_* \mathbf{L}\pi^! N \cong N$ hold for any proper birational morphism between smooth varieties. The next lemma encapsulates the key feature of the surface case, which is much less common for birational morphisms in higher dimension.

Lemma 4.4. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces. Then for any sheaf M on X , we have $R^i \pi_* M = 0$ for $i \geq 2$. In particular, any quotient of a π_* -acyclic sheaf is π_* -acyclic.*

Proof. Indeed, the fibers of π are 1-dimensional. \square

Since $\pi^!$ is a twist of π^* , we find that sheaves of homological dimension 1 are also acyclic for $\pi^!$. Thus if M is a sheaf of homological dimension 1, we also have a natural isomorphism

$$M \cong \pi_* \pi^! M \quad (4.12)$$

Either this or the isomorphism $\pi_* \pi^* M \cong M$ induces by adjointness a natural morphism

$$\pi^* M \rightarrow \pi^! M, \quad (4.13)$$

which will be an isomorphism outside the exceptional locus of π .

Definition 2. Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, and let M be a coherent sheaf on Y with homological dimension ≤ 1 . Then the *minimal lift* of M is the image $\pi^! M$ of the natural map $\pi^* M \rightarrow \pi^! M$.

Remark. In the theory of local systems, there is a functor “middle extension”, the image of the natural transformation $j_! \rightarrow j_*$ where j is an open immersion [11]. Since Hitchin systems are relaxations of moduli spaces of differential equations, it is likely that this is more than just a formal analogy. Also, the middle extension is most naturally defined on perverse sheaves; is there an analogous notion in the present setting?

To justify the name, we have the following quasi-universal property.

Proposition 4.5. *Let N be a sheaf of homological dimension ≤ 1 on Y , and suppose M is a π_* -acyclic sheaf on X with direct image N . Then M has a natural subquotient isomorphic to $\pi^! N$.*

Proof. The isomorphism $\pi_* M \cong N$ and its inverse induce by adjointness natural maps

$$\pi^* N \rightarrow M \rightarrow \pi^! N. \quad (4.14)$$

It remains only to show that the composition agrees with the canonical map $\pi^* N \rightarrow \pi^! N$. Since we obtained the factors by adjointness, it follows that the composition

$$N \rightarrow \pi_* \pi^* N \rightarrow \pi_* M \rightarrow \pi_* \pi^! N \rightarrow N \quad (4.15)$$

is the identity, which implies that the map $\pi^* N \rightarrow \pi^! N$ is adjoint to the inverse of the canonical morphism $N \rightarrow \pi_* \pi^* N$, as required. \square

Of course, we also want to know that $\pi^! M$ itself satisfies this!

Proposition 4.6. *Let M be a sheaf of homological dimension ≤ 1 on Y . Then $\pi^{*!}M$ is π_* -acyclic, and there is a natural isomorphism $\pi_*\pi^{*!}M \cong M$.*

Proof. Since $\pi^{*!}M$ is a quotient of the π_* -acyclic sheaf π^*M , it is certainly π_* -acyclic. Moreover, the composition

$$M \cong \pi_*\pi^*M \rightarrow \pi_*\pi^{*!}M \rightarrow \pi_*\pi^!M \cong M \quad (4.16)$$

is the identity, and the map

$$\pi_*\pi^{*!}M \rightarrow \pi_*\pi^!M \quad (4.17)$$

is injective. The claim follows. \square

The minimal lift behaves well under composition of birational morphisms.

Proposition 4.7. *Let $\pi_1 : X \rightarrow Y$, $\pi_2 : Y \rightarrow Z$ be birational morphisms of algebraic surfaces. Then for any coherent sheaf M on Z with homological dimension ≤ 1 ,*

$$(\pi_2 \circ \pi_1)^{*!}M \cong \pi_1^{*!}\pi_2^{*!}M. \quad (4.18)$$

Proof. Consider the composition

$$\pi_1^*\pi_2^*M \rightarrow \pi_1^*\pi_2^{*!}M \rightarrow \pi_1^{*!}\pi_2^{*!}M \rightarrow \pi_1^!\pi_2^{*!}M \rightarrow \pi_1^!\pi_2^!M. \quad (4.19)$$

The second map is surjective and the third map injective by definition of $\pi_1^{*!}$, and the first map is surjective since π_1^* is right exact. It remains only to show that the fourth map is injective, for which it will suffice to show that $\pi_2^!M/\pi_2^{*!}M$ has homological dimension 1. If not, then the quotient contains the structure sheaf of a point, and thus has nontrivial direct image under π_2 , contradicting the fact that $\pi_2^{*!}$ and $\pi_2^!M$ are π_{2*} -acyclic sheaves with isomorphic direct images.

It follows that $\pi_1^{*!}\pi_2^{*!}M$ is the image of the natural map

$$(\pi_2 \circ \pi_1)^*M \cong \pi_1^*\pi_2^*M \rightarrow \pi_1^!\pi_2^!M \cong (\pi_2 \circ \pi_1)^!M \quad (4.20)$$

as required. \square

We also want to compare the minimal lift to the lifting operation in the theory of Hitchin systems.

Proposition 4.8. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, and let $C \subset X$ be a curve intersecting the exceptional locus of π transversely. Then for any torsion-free coherent sheaf M on C , viewed as a sheaf on X ,*

$$\pi^{*!}\pi_*M \cong M. \quad (4.21)$$

Proof. The transversality assumption implies that $\pi|_C : C \rightarrow Y$ has 0-dimensional fibers, so M is certainly π_* -acyclic. Moreover, if we twist M by the inverse of a sufficiently ample bundle, we can arrange that M has no global sections, so that the same applies to its direct image. But then π_*M certainly cannot have a 0-dimensional subsheaf, so that π_*M has homological dimension ≤ 1 .

It follows that M has $\pi^{*!}\pi_*M$ as a subquotient. Moreover, they are isomorphic away from the exceptional locus, so the residual sub- and quotient sheaves of M are supported there. Transversality then makes both sheaves 0-dimensional, so that the torsion-free hypothesis makes the subsheaf trivial. We thus have a short exact sequence

$$0 \rightarrow \pi^{*!}\pi_*M \rightarrow M \rightarrow T \rightarrow 0 \quad (4.22)$$

where T is 0-dimensional. As before, we compute $\pi_*T = 0$ from the long exact sequence, and thus $T = 0$ as required. \square

Remark. In particular, if the sheaf N on Y is an invertible sheaf on a smooth curve (or simply the image of such a curve under a morphism), then $\pi^{*!}N$ is the corresponding sheaf on the strict transform.

Another important example of minimal lifts is the following.

Proposition 4.9. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, and suppose that C is a curve on Y . Then $\pi^{*!}\mathcal{O}_C \cong \mathcal{O}_{C'}$ for some curve C' . If X, Y, π are Poisson and C is the anticanonical curve on Y , then C' is the anticanonical curve on X .*

Proof. It suffices to consider the case that π is monoidal, with center p . If $p \notin C$, then $\pi^*\mathcal{O}_C \cong \pi^!\mathcal{O}_C \cong \mathcal{O}_{\pi^{-1}C}$, so the result is immediate. If $p \in C$, then over the local ring at p , \mathcal{O}_C has a minimal resolution

$$0 \rightarrow (\mathcal{O}_Y)_p \rightarrow (\mathcal{O}_Y)_p \rightarrow \mathcal{O}_C \otimes (\mathcal{O}_Y)_p \rightarrow 0, \quad (4.23)$$

from which we can compute

$$\pi^{*!}\mathcal{O}_C \cong \mathcal{O}_{\pi^*C-e}. \quad (4.24)$$

In particular, in the Poisson case, we must have $p \in C_\alpha$, and $\pi^*C_\alpha - e$ is indeed the anticanonical divisor on X . \square

5 Invisible sheaves

Since $\pi^{*!}M$ is naturally a quotient of π^*M and a subsheaf of $\pi^!M$, and adjointness makes the latter sheaves easy to deal with, this suggests that an investigation of the corresponding kernel and cokernel might be rewarding.

A key property of those sheaves is that they are in a sense invisible to π_* . To be precise, as we will see, they satisfy the following definition. We again restrict our attention to smooth surfaces over algebraically closed fields.

Definition 3. Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces. A coherent sheaf E on X is π -invisible if it is acyclic with trivial direct image.

Remark. In other words, E is π -invisible iff $\mathbf{R}\pi_*E = 0$.

We will omit π from the notation when it is clear from context. Note that an invisible sheaf is necessarily supported (set-theoretically) on the exceptional locus, since π is an isomorphism elsewhere. Also, if E had a 0-dimensional subsheaf, that subsheaf would have nontrivial direct image; we thus find that invisible sheaves have homological dimension 1.

Proposition 5.1. *Suppose M^\bullet is a complex of sheaves on X such that $\mathbf{R}\pi_*M^\bullet = 0$. Then every homology sheaf of M is invisible.*

Proof. Since π has ≤ 1 -dimensional fibers, $R^p\pi_* = 0$ for $p \geq 2$. Thus the hypercohomology spectral sequence

$$R^p\pi_*(H^q(M^\bullet)) \Rightarrow R^{p+q}\pi_*M^\bullet \quad (5.1)$$

collapses at the E_2 page. Since the limit of the spectral sequence is 0, every term on the E_2 page is 0, and thus

$$\pi_*(H^q(M^\bullet)) = R^1\pi_*(H^q(M^\bullet)) = 0. \quad (5.2)$$

In other words, $H^q(M^\bullet)$ is invisible for all q . \square

The most important special case of this for our purposes is the following.

Corollary 5.2. *Suppose $f : M \rightarrow N$ is a morphism of π_* -acyclic sheaves on X such that $\pi_* f$ is an isomorphism. Then $\ker(f)$ and $\operatorname{coker}(f)$ are invisible. Conversely, if f has invisible kernel and cokernel, then $\pi_* f$ is an isomorphism; moreover, $\operatorname{im}(f)$ is also π_* -acyclic with $\pi_* M \cong \pi_* \operatorname{im}(f) \cong \pi_* N$.*

Proof. The first claim is immediate from the proposition; for the second, factoring f through its image reduces to the cases that f is injective or surjective. But since the cokernel/kernel is invisible, we obtain an isomorphism between the remaining two derived direct images, and composing find that $\pi_* f$ is an isomorphism as required. \square

In particular, this implies that the sheaves $\ker(\pi^* M \rightarrow \pi^{*!} M)$ and $\pi^! M / \pi^{*!} M$ are invisible, as we indicated above.

Proposition 5.3. *The category of π -invisible sheaves is closed under taking kernels, cokernels and extensions. In particular, it is an abelian category.*

Proof. The claim for kernels and cokernels follows from Corollary 5.2. For extensions

$$0 \rightarrow E \rightarrow E' \rightarrow E'' \rightarrow 0, \quad (5.3)$$

the long exact sequence corresponding to $\mathbf{R}\pi_*$ immediately tells us that if two of the sheaves are invisible, then so is the third. \square

Another source of invisible sheaves is the following.

Proposition 5.4. *Suppose $f : M \rightarrow N$ is a morphism of sheaves on X such that M is π_* -acyclic, and $\pi_* N$ has homological dimension ≤ 1 . If f vanishes outside the exceptional locus of π , then $\operatorname{im}(f)$ is invisible, and $\pi_* f = 0$.*

Proof. Since M is π_* -acyclic, so is its quotient $\operatorname{im}(f)$. Since f vanishes outside the exceptional locus, $\pi_* \operatorname{im}(f)$ is 0-dimensional; but $\pi_* N$ has no 0-dimensional subsheaf. \square

Invisible sheaves also interact nicely with the lifting operations.

Proposition 5.5. *Let E be an invisible sheaf on X , and M a sheaf on Y with homological dimension ≤ 1 . Then*

$$\mathbf{R}\operatorname{Hom}(\pi^* M, E) = \mathbf{R}\operatorname{Hom}(E, \pi^! M) = 0. \quad (5.4)$$

Proof. This follows immediately from adjointness:

$$\begin{aligned} \mathbf{R}\operatorname{Hom}(\pi^* M, E) &\cong \mathbf{R}\operatorname{Hom}(M, \mathbf{R}\pi_* E) = 0, \\ \mathbf{R}\operatorname{Hom}(E, \pi^! M) &\cong \mathbf{R}\operatorname{Hom}(\mathbf{R}\pi_* E, M) = 0. \end{aligned}$$

\square

Proposition 5.6. *Let E and M be as before. Then*

$$\operatorname{Hom}(\pi^{*!} M, E) = \operatorname{Hom}(E, \pi^{*!} M) = 0. \quad (5.5)$$

Proof. Indeed,

$$\begin{aligned} \operatorname{Hom}(E, \pi^{*!} M) &\subset \operatorname{Hom}(E, \pi^! M) = 0, \\ \operatorname{Hom}(\pi^{*!} M, E) &\subset \operatorname{Hom}(\pi^* M, E) = 0. \end{aligned}$$

\square

Remark. We will see below that this actually characterizes those sheaves which are minimal lifts.

Invisible sheaves behave well under lifts and direct images.

Lemma 5.7. *Suppose $\pi : X \rightarrow Y$ and $\phi : Y \rightarrow Z$ are birational morphisms of algebraic surfaces. If a sheaf E on Y is ϕ -invisible, then π^*E , $\pi^!E$, and $\pi^{*!}E$ are $\phi \circ \pi$ -invisible. If a sheaf E on X is $\phi \circ \pi$ -invisible, then it is π_* -acyclic and π_*E is ϕ -invisible. Moreover, π_* and π^* take projective objects (of the category of invisible sheaves) to projective objects, and π_* and $\pi^!$ take injective objects to injective objects.*

Proof. First suppose E is ϕ_* -invisible, and let E' be any of the three lifts of E to X . Then we have a natural isomorphism $\mathbf{R}\pi_*E' \cong E$, and thus

$$\mathbf{R}(\phi \circ \pi)_*E' \cong \mathbf{R}\phi_*\mathbf{R}\pi_*E' \cong \mathbf{R}\phi_*E = 0 \quad (5.6)$$

as required.

Now, let E be a $\phi \circ \pi$ -invisible sheaf on X . We again have

$$\mathbf{R}\phi_*\mathbf{R}\pi_*E = 0. \quad (5.7)$$

Since ϕ and π have ≤ 1 -dimensional fibers, the corresponding spectral sequence collapses at the E_2 page, and thus

$$R^p\phi_*R^q\pi_*E = 0 \quad (5.8)$$

for $p, q \in \{0, 1\}$. In particular, π_*E is invisible, and we need only show that E is π_* -acyclic. But since $R^1\pi_*E$ is supported on the indeterminacy locus of π^{-1} , the only way it can have trivial direct image is to be 0.

The claims about projective and injective objects follow from adjointness. For instance if E is projective in the category of $\pi \circ \phi$ -invisible sheaves, and E' is any ϕ -invisible sheaf, then

$$\mathrm{Ext}^{p+1}(\pi_*E, E') \cong \mathrm{Ext}^{p+1}(E, \pi^!E') = 0. \quad (5.9)$$

□

Thus the atomic case (monoidal transformations blowing up a single point) will be particularly useful. Here, we can completely characterize the invisible sheaves.

Lemma 5.8. *Suppose that $\pi : X \rightarrow Y$ is a monoidal transformation, blowing up the point $p \in Y$ to the exceptional line $e \subset X$. Then the functor*

$$E \mapsto \mathrm{Hom}(\mathcal{O}_e(-1), E) \quad (5.10)$$

establishes an equivalence between the category of π -invisible sheaves and the category of finite-dimensional vector spaces over \bar{k} .

Proof. Let D be an effective divisor on Y that has multiplicity 1 at p . Then we obtain a short exact sequence

$$0 \rightarrow \mathcal{O}_e(-1) \rightarrow \pi^*\mathcal{O}_D \rightarrow \pi^{*!}\mathcal{O}_D \rightarrow 0. \quad (5.11)$$

(Use the minimal resolution of \mathcal{O}_D over the local ring at p to compute $\pi^{*!}\mathcal{O}_D$.) We thus find

$$R^p\mathrm{Hom}(\mathcal{O}_e(-1), E) \cong R^{p+1}\mathrm{Hom}(\pi^{*!}\mathcal{O}_D, E) \cong R^{p+1}\mathrm{Hom}(\mathcal{O}_{\tilde{D}}, E), \quad (5.12)$$

where \tilde{D} is the strict transform of D . Since \tilde{D} is transverse to the exceptional line, it is transverse to the support of E , and thus the only nonvanishing Ext group is in degree 1. We thus conclude that for any invisible sheaf,

$$R^p \text{Hom}(\mathcal{O}_e(-1), E) = 0 \quad (5.13)$$

for $p > 0$; in other words, $\mathcal{O}_e(-1)$ is a projective object in the category of invisible sheaves.

Since $\mathcal{O}_e(-1)$ is coherent, with endomorphism ring k , it remains only to show that it generates the category, since then the desired equivalence follows by Morita theory. In other words, we need to show that if $\text{Hom}(\mathcal{O}_e(-1), E) = 0$, then $E = 0$. Since \tilde{D} is transverse to the exceptional locus, we have a short exact sequence

$$0 \rightarrow E \rightarrow E \otimes \mathcal{L}(\tilde{D}) \rightarrow T \rightarrow 0 \quad (5.14)$$

for some 0-dimensional sheaf T . If $T = 0$, then E has support disjoint from \tilde{D} ; if nontrivial, it would have 0-dimensional support, and thus nontrivial direct image. Thus $T \neq 0$, and we conclude that

$$\chi(E \otimes \mathcal{L}(\tilde{D})) = \chi(T) > 0, \quad (5.15)$$

so that $E \otimes \mathcal{L}(\tilde{D})$ has global sections, and its direct image is thus a nontrivial 0-dimensional sheaf. It follows from the next lemma that

$$\text{Hom}(\mathcal{O}_e(-1), E) \cong \text{Hom}(\mathcal{O}_e, E \otimes \mathcal{L}(\tilde{D})) \neq 0, \quad (5.16)$$

as required. \square

Lemma 5.9. *Let $\pi : X \rightarrow Y$ be the blowup in the closed point p of Y , with exceptional line e , and let M be a coherent sheaf on X of homological dimension ≤ 1 . If M is not π_* -acyclic, then $\text{Hom}(M, \mathcal{O}_e(-2)) \neq 0$, while if $\pi_* M$ has homological dimension 2, then $\text{Hom}(\mathcal{O}_e, M) \neq 0$.*

Proof. Since π is an isomorphism away from e and p , the only way that $\pi_* M$ can have homological dimension 2 is if it has a subsheaf isomorphic to \mathcal{O}_p . We can compute

$$\mathbf{R} \text{Hom}(\mathcal{O}_p, \mathbf{R} \pi_* M) \cong \mathbf{R} \text{Hom}(\mathbf{L} \pi^* \mathcal{O}_p, M) \quad (5.17)$$

so in particular

$$\text{Hom}(\mathcal{O}_p, \pi_* M) \cong \text{Hom}(\pi^* \mathcal{O}_p, M). \quad (5.18)$$

A simple calculation in the local ring at p gives

$$L_1 \pi^* \mathcal{O}_p \cong \mathcal{O}_e(-1), \quad \pi^* \mathcal{O}_p \cong \mathcal{O}_e, \quad (5.19)$$

giving

$$\text{Hom}(\mathcal{O}_p, \pi_* M) \cong \text{Hom}(\mathcal{O}_e, M), \quad (5.20)$$

implying the desired result.

Similarly, if M is not π_* -acyclic, then $R^1 \pi_* M$ is a nontrivial sheaf supported at p , so has a morphism to \mathcal{O}_p . We compute

$$\mathbf{R} \text{Hom}(\mathbf{R} \pi_* M, \mathcal{O}_p) \cong \mathbf{R} \text{Hom}(M, \mathbf{L} \pi^! \mathcal{O}_p) \quad (5.21)$$

which in degree -1 gives

$$\text{Hom}(R^1 \pi_* M, \mathcal{O}_p) \cong \text{Hom}(M, L_1 \pi^! \mathcal{O}_p) \cong \text{Hom}(M, \mathcal{O}_e(-2)). \quad (5.22)$$

\square

Corollary 5.10. *Let $\pi : X \rightarrow Y$ be the blowup in a single point $p \in Y$, with exceptional line e , and let M be a sheaf of homological dimension ≤ 1 on X . If $\text{Hom}(\mathcal{O}_e(-1), M) = \text{Hom}(M, \mathcal{O}_e(-1)) = 0$, then $M \cong \pi^{*!}\pi_*M$.*

Proof. We first observe that M is π_* -acyclic with π_*M of homological dimension ≤ 1 . Indeed, we would otherwise have a nonzero map $\mathcal{O}_e \rightarrow M$ or $M \rightarrow \mathcal{O}_e(-2)$, which we could compose with any nonzero map $\mathcal{O}_e(-1) \rightarrow \mathcal{O}_e$ or $\mathcal{O}_e(-2) \rightarrow \mathcal{O}_e(-1)$. The second composition is necessarily nonzero by injectivity of the second map; the first composition could only be zero if the original map had 0-dimensional image.

In particular, we find that we have natural maps

$$\pi^*\pi_*M \rightarrow M \rightarrow \pi^!\pi_*M. \quad (5.23)$$

The first map is surjective, since its cokernel is invisible, and the hypotheses imply that M has no nonzero maps to invisible sheaves. Similarly, the second map is injective since it has invisible kernel. In other words, M is the image of the natural map $\pi^*\pi_*M \rightarrow \pi^!\pi_*M$ as required. \square

For a more general birational morphism, the exceptional locus is still a union of finitely many smooth rational curves, which we call the *exceptional components* of π .

Lemma 5.11. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, and let M be a nonzero coherent sheaf on X such that $\pi_*M = 0$. Then there is some exceptional component f such that $\text{Hom}(M, \mathcal{O}_f(-1)) \neq 0$.*

Proof. If π is the blowup in a point $p \in Y$, then either $R^1\pi_*M \neq 0$, in which case it maps to $\mathcal{O}_e(-2)$ and thus $\mathcal{O}_e(-1)$, or $R^1\pi_*M = 0$, in which case it is invisible, and thus is isomorphic to $\mathcal{O}_e(-1)^r$.

Otherwise, we can factor $\pi = \pi_1 \circ \pi_2$ such that π_1 blows up a single point. If $\text{Hom}(M, \mathcal{O}_e(-1)) \neq 0$, where e is the exceptional locus of π_1 , then we are done. Otherwise, M is π_{1*} -acyclic, and $\pi_{2*}(\pi_{1*}M) \cong \pi_*M = 0$. Moreover, we have a short exact sequence of the form

$$0 \rightarrow \mathcal{O}_e(-1)^r \rightarrow \pi_1^*\pi_{1*}M \rightarrow M \rightarrow 0. \quad (5.24)$$

Since $\text{Hom}(\mathcal{O}_e(-1), \mathcal{O}_f(-1))$ for $f \neq e$, it will suffice to show that

$$\text{Hom}(\pi_1^*\pi_{1*}M, \mathcal{O}_f(-1)) \neq 0 \quad (5.25)$$

for some exceptional component $f \neq e$. But

$$\text{Hom}(\pi_1^*\pi_{1*}M, \mathcal{O}_f(-1)) \cong \text{Hom}(\pi_{1*}M, \pi_{1*}\mathcal{O}_f(-1)) \cong \text{Hom}(\pi_{1*}M, \mathcal{O}_{\pi_1(f)}(-1)) \quad (5.26)$$

Every exceptional component of π_2 is the image of some $f \neq e$, and thus at least one of the latter groups is nonzero. \square

Corollary 5.12. *Any invisible sheaf has a quotient of the form $\mathcal{O}_f(-1)$ for some exceptional component f .*

Proof. By the lemma, any invisible sheaf has a nonzero morphism to some $\mathcal{O}_f(-1)$. The image of such a morphism has the form $\mathcal{O}_f(-d)$ for some $d > 1$. Since it is a quotient of an invisible, thus π_* -acyclic, sheaf, it must be π_* -acyclic. But then

$$H^1(\mathcal{O}_f(-d)) = H^1(\pi_*\mathcal{O}_f(-d)) = 0, \quad (5.27)$$

since it has 0-dimensional direct image. It follows that $d = 1$. \square

Corollary 5.13. *The category of invisible sheaves is artinian.*

Proof. Let $E_1 \supsetneq E_2 \supsetneq E_3 \supsetneq \cdots$ be a descending chain of invisible sheaves. Each quotient E_i/E_{i+1} is a nontrivial invisible sheaf, so surjects on some $\mathcal{O}_f(-1)$. In particular, relative to any very ample bundle on X , $\deg(c_1(E_i))$ is a strictly decreasing sequence of nonnegative integers. \square

Corollary 5.14. *Any invisible sheaf admits a filtration in which the successive quotients are all of the form $\mathcal{O}_f(-1)$ with f an exceptional component.*

Proof. Any nontrivial invisible sheaf has a nontrivial map to some $\mathcal{O}_f(-1)$, and the kernel is invisible. Iterating gives a descending chain of invisible sheaves, which must eventually reach 0. \square

This gives us an alternate characterization of minimal lifts.

Theorem 5.15. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, and suppose that M is a sheaf on X of homological dimension ≤ 1 . If $\text{Hom}(M, \mathcal{O}_f(-1)) = 0$ for all exceptional components f , then M is π_* -acyclic and π -globally generated. If $\text{Hom}(\mathcal{O}_f(-1), M) = 0$ for all exceptional components f , then π_*M has homological dimension ≤ 1 . Moreover, M is a minimal lift iff $\text{Hom}(\mathcal{O}_f(-1), M) = \text{Hom}(M, \mathcal{O}_f(-1)) = 0$ for all exceptional components f .*

Proof. Suppose that $\text{Hom}(M, \mathcal{O}_f(-1)) = 0$ for all exceptional components, and consider the natural map

$$\pi^* \pi_* M \rightarrow M. \quad (5.28)$$

The direct image of this map is an isomorphism, and thus the usual spectral sequence shows that the cokernel has trivial direct image. It follows that if the cokernel is nonzero, then it maps to $\mathcal{O}_f(-1)$ for some f , giving a corresponding map on M . We thus conclude that M is π -globally generated, and thus π_* -acyclic (as a quotient of the π_* -acyclic sheaf $\pi^* \pi_* M$).

Next, note that if $\text{Hom}(\mathcal{O}_f(-1), M) = 0$ for all exceptional components, then $\text{Hom}(E, M) = 0$ for all invisible sheaves E . So it will suffice to show that if π_*M has homological dimension 2, then $\text{Hom}(E, M) \neq 0$ for some invisible E . Factor $\pi = \pi_1 \circ \pi_2$ with π_1 monoidal. If $\pi_{1*}M$ has homological dimension 2, then $\text{Hom}(\mathcal{O}_e, M) \neq 0$, and thus $\text{Hom}(\mathcal{O}_e(-1), M) \neq 0$. Otherwise, we have $\text{Hom}(E, \pi_{1*}M) \neq 0$ for some π_2 -invisible E , and thus $\text{Hom}(\pi_1^*E, M) \neq 0$. Since π_1^*E is π -invisible, the claim follows.

Now, suppose that both conditions hold. Then in the composition

$$\pi^* \pi_* M \rightarrow M \rightarrow \pi^! \pi_* M \quad (5.29)$$

we have already shown the first map to be surjective, and the second map has invisible kernel, so must be injective. That $M \cong \pi^{*!} \pi_* M$ follows. \square

Given an exceptional component f , let f^\vee denote the linear combination of exceptional components such that $f^\vee \cdot g = -\delta_{fg}$ for exceptional components g . (Note that although X is not assumed projective, it still has a well-behaved intersection theory, at least where exceptional divisors are concerned.) Equivalently,

$$\mathcal{O}_g \otimes \mathcal{L}(f^\vee) \cong \mathcal{O}_g(-\delta_{fg}). \quad (5.30)$$

By induction on a factorization of π into monoidal transformations, we find that f^\vee is effective. Define a sheaf P_f by the exact sequence

$$0 \rightarrow P_f \rightarrow \pi^* \pi_* \mathcal{L}(-f^\vee) \rightarrow \mathcal{L}(-f^\vee). \quad (5.31)$$

Lemma 5.16. *The sheaves P_f are projective objects in the category of π -invisible sheaves. More precisely, P_f is the projective cover of $\mathcal{O}_f(-1)$.*

Proof. Since

$$\mathrm{Hom}(\mathcal{L}(-f^\vee), \mathcal{O}_g(-1)) \cong H^0(\mathcal{O}_g(-1 - \delta_{fg})) = 0, \quad (5.32)$$

it follows that $\mathcal{L}(-f^\vee)$ is π_* -acyclic and π -globally generated, so that the exact sequence defining P_f extends to a short exact sequence. In addition, we have

$$\pi_* \mathcal{L}(-f^\vee) \subset \pi_* \mathcal{O}_X \cong \mathcal{O}_Y, \quad (5.33)$$

and subsheaves of locally free sheaves have homological dimension ≤ 1 .

Thus P_f is an invisible sheaf, and we have

$$R^p \mathrm{Hom}(P_f, \mathcal{O}_g(-1)) \cong R^{p+1} \mathrm{Hom}(\mathcal{L}(-f^\vee), \mathcal{O}_g(-1)) \cong H^{p+1}(\mathcal{O}_g(-1 - \delta_{fg})) \quad (5.34)$$

In particular, $R^p \mathrm{Hom}(P_f, \mathcal{O}_g(-1)) = 0$ for $p > 0$; since every invisible sheaf is an extension of sheaves $\mathcal{O}_g(-1)$, it follows that P_f is projective. Moreover,

$$\dim H^1(\mathcal{O}_g(-1 - \delta_{fg})) = \delta_{fg}, \quad (5.35)$$

and thus P_f has a unique map to $\mathcal{O}_f(-1)$, and maps to no other component; it is thus the projective cover of $\mathcal{O}_f(-1)$. \square

Remark. Taking the direct image of the short exact sequence

$$0 \rightarrow \mathcal{L}(-f^\vee) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{f^\vee} \rightarrow 0 \quad (5.36)$$

gives

$$0 \rightarrow \pi_* \mathcal{L}(-f^\vee) \rightarrow \mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_{f^\vee} \rightarrow 0, \quad (5.37)$$

so that $\pi_* \mathcal{L}(-f^\vee)$ is the ideal sheaf of the 0-dimensional subscheme $\pi(f^\vee) \subset Y$, and

$$P_f \cong L_1 \pi^* \pi_* \mathcal{O}_{f^\vee} \cong L_1 \pi^* \mathcal{O}_{\pi(f^\vee)}. \quad (5.38)$$

Since the category of π -invisible sheaves has finitely many irreducible objects, and every irreducible object has a projective cover, it is isomorphic to a module category. More precisely, we have the following, by a straightforward application of Morita theory.

Theorem 5.17. *There is a natural equivalence between the category of π -invisible sheaves and the category of finitely generated modules over the finite-dimensional k -algebra $\mathrm{End}(\bigoplus_f P_f)$, given by*

$$E \mapsto \mathrm{Hom}(\bigoplus_f P_f, E). \quad (5.39)$$

Moreover, the Yoneda Ext groups in this category agree with the Ext groups in the category of sheaves.

Proof. The main claim follows from the fact that $\bigoplus_f P_f$ is a progenerator of the category (it is coherent, projective, and generates the category).

By using a projective resolution to compute the Yoneda Ext groups, we see that the remaining claim reduces to showing that the group $\mathrm{Ext}_{\mathrm{Coh}(X)}^p(P, E)$ vanishes for $p > 0$, P projective, and E invisible. But this is true for any projective object of the form P_f , and thus for direct summands of direct sums of such sheaves. \square

Since the canonical map $\pi^*\mathcal{O}_Y \rightarrow \pi^!\mathcal{O}_Y$ is injective (it has invisible kernel, but \mathcal{O}_X is torsion-free), it gives rise to a canonical divisor e_π representing $K_X - \pi^*K_Y$, with e_π supported on the exceptional locus.

Corollary 5.18. *Every invisible sheaf is scheme-theoretically supported on a subscheme of e_π .*

Proof. Every invisible sheaf is a quotient of a sum of sheaves P_f , so it suffices to consider those. Then P_f is contained in the kernel E of the natural map

$$\pi^*\pi_*\mathcal{L}(-f^\vee) \rightarrow \pi^!\pi_*\mathcal{L}(-f^\vee). \quad (5.40)$$

Starting with a locally free resolution

$$0 \rightarrow V \rightarrow W \rightarrow \pi_*\mathcal{L}(-f^\vee) \rightarrow 0, \quad (5.41)$$

we can use the snake lemma to obtain an exact sequence

$$0 \rightarrow E \rightarrow \pi^!V/\pi^*V \rightarrow \pi^!W/\pi^*W. \quad (5.42)$$

Since

$$\pi^!V/\pi^*V \cong (\pi^!\mathcal{O}_Y/\pi^*\mathcal{O}_Y) \otimes \pi^*V \cong \mathcal{O}_{e_\pi} \otimes \mathcal{L}(e_\pi) \otimes \pi^*V, \quad (5.43)$$

the result follows. \square

Remark. The invisible sheaf $\pi^!\mathcal{O}_Y/\pi^*\mathcal{O}_Y \cong \mathcal{O}_{e_\pi} \otimes \mathcal{L}(e_\pi)$ shows that this bound on the support is tight.

The injective objects in the category of invisible sheaves can also be identified, most easily using the following duality, a special case of a standard duality for sheaves of homological dimension 1 and codimension 1 support.

Lemma 5.19. *If E is an invisible sheaf, then so is $\mathcal{E}xt^1(E, \omega_X)$, and $\mathcal{E}xt^1(\mathcal{E}xt^1(E, \omega_X), \omega_X) \cong E$.*

Proof. Since E has 1-dimensional support and homological dimension 1,

$$\mathcal{H}om(E, \omega_X) = \mathcal{E}xt^2(E, \omega_X) = 0, \quad (5.44)$$

and thus $\mathbf{R}\mathcal{H}om(E, \omega_X)$ is concentrated in degree 1. Since $\omega_X = \pi^!\omega_Y$, we have

$$\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(E, \omega_X) \cong \mathbf{R}\mathcal{H}om(\mathbf{R}\pi_*E, \omega_Y) = 0, \quad (5.45)$$

and thus $\mathbf{R}\pi_*\mathcal{E}xt^1(E, \omega_X) = 0$ as required.

That this operation is a duality follows by considering how it acts on a locally free resolution $0 \rightarrow V \rightarrow W \rightarrow E \rightarrow 0$. \square

Remark. This implies that the k -algebra associated to π above is isomorphic to its opposite.

In particular, we can define injective objects

$$I_f := \mathcal{E}xt^1(P_f, \omega_X) \cong \mathcal{E}xt^1(\pi^*\pi_*\mathcal{L}(-f^\vee), \omega_X) \quad (5.46)$$

and find that I_f is the injective hull of $\mathcal{O}_f(-1)$. Note that the sheaves $\mathcal{O}_f(-1)$ are self-dual: considerations of support show that the dual of $\mathcal{O}_f(-1)$ has the form $\mathcal{O}_f(-d)$, and $\mathcal{O}_f(-1)$ is the only invisible possibility. Another self-dual invisible sheaf is the sheaf $\pi^!\omega_Y/\pi^*\omega_Y$: the dual is most naturally expressed as $\pi^!\mathcal{O}_Y/\pi^*\mathcal{O}_Y$, but twisting by the pullback of a line bundle has no effect on an invisible sheaf.

6 Invisible sheaves and minimal lifts

In the case of direct images of line bundles on smooth curves, the purpose of lifting is to make the lifted sheaf disjoint from the anticanonical curve. This is of course not possible for sheaves of positive rank, but we can still hope to make the restriction to the anticanonical curve simpler. Thus we would like to understand how the restriction of the minimal lift is related to the original restriction. This is not quite the correct question, as it turns out: it turns out to be easier to twist by a line bundle and consider

$$M \otimes \mathcal{O}_{C_\alpha} \otimes \mathcal{L}(C_\alpha). \quad (6.1)$$

The standard resolution

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{L}(C_\alpha) \rightarrow \mathcal{O}_{C_\alpha} \otimes \mathcal{L}(C_\alpha) \rightarrow 0 \quad (6.2)$$

shows that

$$\mathcal{T}or_p(M, \mathcal{O}_{C_\alpha} \otimes \mathcal{L}(C_\alpha)) \cong \mathcal{E}xt_Y^{1-p}(\mathcal{O}_{C_\alpha}, M). \quad (6.3)$$

Thus more generally we want to understand how $\mathcal{E}xt_Y^*(\pi^*M, \pi^*N)$ and $\mathcal{E}xt_Y^*(M, N)$ are related.

Lemma 6.1. *Let $\pi : X \rightarrow Y$ be a birational morphism of smooth surfaces, and suppose M, N are sheaves on Y with homological dimension ≤ 1 . Then there is an isomorphism*

$$\mathcal{H}om_Y(\pi^*M, \pi^*N) \cong \mathcal{H}om_Y(M, N) \quad (6.4)$$

and an exact sequence

$$0 \rightarrow \mathcal{E}xt_Y^1(\pi^*M, \pi^*N) \rightarrow \mathcal{E}xt_Y^1(M, N) \rightarrow \mathcal{H}om_Y(E_1, E_2) \rightarrow \mathcal{E}xt_Y^2(\pi^*M, \pi^*N) \rightarrow 0, \quad (6.5)$$

where E_1 and E_2 are the π -invisible sheaves fitting into the short exact sequences

$$0 \rightarrow E_1 \rightarrow \pi^*M \rightarrow \pi^{*!}M \rightarrow 0 \quad (6.6)$$

$$0 \rightarrow \pi^{*!}N \rightarrow \pi^!N \rightarrow E_2 \rightarrow 0. \quad (6.7)$$

Proof. Using the E_1 exact sequence, we obtain the long exact sequence

$$\cdots \rightarrow R^p\mathcal{H}om_Y(\pi^{*!}M, \pi^{*!}N) \rightarrow R^p\mathcal{H}om_Y(\pi^*M, \pi^{*!}N) \rightarrow R^p\mathcal{H}om_Y(E_1, \pi^{*!}N) \rightarrow \cdots \quad (6.8)$$

Moreover, $R^p\mathcal{H}om_Y(\pi^*M, \pi^{*!}N) \cong R^p\mathcal{H}om_Y(M, N)$ by adjointness.

Similarly, since $R^p\mathcal{H}om_Y(E_1, \pi^!N) = 0$, the E_2 exact sequence gives isomorphisms

$$R^p\mathcal{H}om_Y(E_1, E_2) \cong R^{p+1}\mathcal{H}om_Y(E_1, \pi^{*!}N). \quad (6.9)$$

Combining these gives a long exact sequence

$$\cdots \rightarrow R^p\mathcal{H}om_Y(\pi^{*!}M, \pi^{*!}N) \rightarrow R^p\mathcal{H}om_Y(M, N) \rightarrow R^{p-1}\mathcal{H}om_Y(E_1, E_2) \rightarrow \cdots, \quad (6.10)$$

from which the claim follows immediately. Note that $\mathcal{E}xt_Y^2(M, N) = 0$ since M has homological dimension ≤ 1 by assumption. \square

Remark. Similarly, we have an isomorphism

$$\mathcal{H}om(\pi^{*!}M, \pi^{*!}N) \cong \mathcal{H}om(M, N) \quad (6.11)$$

and a long exact sequence

$$\cdots \rightarrow \text{Ext}^{p-2}(E_1, E_2) \rightarrow \text{Ext}^p(\pi^{*!}M, \pi^{*!}N) \rightarrow \text{Ext}^p(M, N) \rightarrow \text{Ext}^{p-1}(E_1, E_2) \rightarrow \cdots \quad (6.12)$$

Corollary 6.2. *Let $\pi : X \rightarrow Y$ be a birational morphism of smooth surfaces, and suppose M is a sheaf on Y with homological dimension ≤ 1 . Then $\text{End}(\pi^* M) \cong \text{End}(M)$. In particular, the minimal lift of a simple sheaf is always simple.*

Corollary 6.3. *Let $\pi : X \rightarrow Y$ be a Poisson birational morphism of Poisson surfaces, and suppose M is a sheaf on Y of homological dimension ≤ 1 . Then*

$$\text{Hom}_Y(\mathcal{O}_{C_{\alpha_X}}, \pi^* M) \cong \text{Hom}_Y(\mathcal{O}_{C_{\alpha_Y}}, M) \quad (6.13)$$

and there is an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{E}xt_Y^1(\mathcal{O}_{C_{\alpha_X}}, \pi^* M) &\rightarrow \mathcal{E}xt_Y^1(\mathcal{O}_{C_{\alpha_Y}}, M) \\ &\rightarrow \text{Hom}_Y(\omega_X / \pi^* \omega_Y, \pi^! M / \pi^* M) \rightarrow \mathcal{E}xt_Y^2(\mathcal{O}_{C_{\alpha_X}}, \pi^* M) \rightarrow 0. \end{aligned}$$

For a monoidal transformation, the invisible sheaves associated to a minimal lift are straightforward to compute.

Proposition 6.4. *Let $\pi : X \rightarrow Y$ be a monoidal transformation, and let M be a sheaf on Y of homological dimension ≤ 1 . Then there is an exact sequence*

$$0 \rightarrow E_1 \rightarrow \pi^* M \rightarrow \pi^! M \rightarrow E_2 \rightarrow 0, \quad (6.14)$$

where we have non-canonical isomorphisms

$$\begin{aligned} E_1 &\cong \text{Ext}^1(\mathcal{O}_p, M) \otimes_k \mathcal{O}_e(-1), \\ E_2 &\cong \text{Hom}_k(\text{Hom}(M, \mathcal{O}_p), \mathcal{O}_e(-1)). \end{aligned}$$

Proof. There certainly is an exact sequence of the given form, since the kernel and cokernel are invisible, thus powers of $\mathcal{O}_e(-1)$. From the exact sequence

$$0 \rightarrow E_1 \rightarrow \pi^* M \rightarrow \pi^* M \rightarrow 0, \quad (6.15)$$

we find

$$\begin{aligned} \text{Hom}(\mathcal{O}_e(-1), E_1) &\cong \text{Hom}(\mathcal{O}_e(-1), \pi^* M) \cong \text{Hom}(\mathcal{O}_e(-2), \pi^! M) \\ &\cong \text{Ext}^1(R^1 \pi_* \mathcal{O}_e(-2), M) \cong \text{Ext}^1(\mathcal{O}_p, M). \end{aligned}$$

Similarly,

$$\text{Hom}(E_2, \mathcal{O}_e(-1)) \cong \text{Hom}(\pi^! M, \mathcal{O}_e(-1)) \cong \text{Hom}(\pi^* M, \mathcal{O}_e) \cong \text{Hom}(M, \mathcal{O}_p). \quad (6.16)$$

□

This has a nice consequence for twists of minimal lifts, which we will generalize below.

Corollary 6.5. *Let $\pi : X \rightarrow Y$ be a monoidal transformation, and M a sheaf on Y of homological dimension ≤ 1 . Then the sheaves $\pi^* M \otimes \mathcal{L}(\pm e)$ are π_* -acyclic, with direct image of homological dimension ≤ 1 . Moreover, we have short exact sequences*

$$0 \rightarrow M \rightarrow \pi_*(\pi^* M \otimes \mathcal{L}(e)) \rightarrow \text{Ext}^1(\mathcal{O}_p, M) \otimes_k \mathcal{O}_p \rightarrow 0, \quad (6.17)$$

and

$$0 \rightarrow \pi_*(\pi^* M \otimes \mathcal{L}(-e)) \rightarrow M \rightarrow \text{Hom}_k(\text{Hom}(M, \mathcal{O}_p), \mathcal{O}_p) \rightarrow 0. \quad (6.18)$$

In other words, $\pi_*(\pi^* M \otimes \mathcal{L}(e))$ is the universal extension of \mathcal{O}_p by M , and $\pi_*(\pi^* M \otimes \mathcal{L}(-e))$ is the kernel of the universal homomorphism from M to \mathcal{O}_p .

Proof. Take the short exact sequences

$$\begin{aligned} 0 \rightarrow E_1 \rightarrow \pi^* M \rightarrow \pi^{*!} M \rightarrow 0 \\ 0 \rightarrow \pi^{*!} M \rightarrow \pi^! M \rightarrow E_2 \rightarrow 0, \end{aligned}$$

twist the first by $\mathcal{L}(e)$ and the second by $\mathcal{L}(-e)$, then take (higher) direct images. Note that after twisting, the middle terms in the short exact sequence are still π_* -acyclic (since they are $\pi^! M$ and $\pi^* M$ respectively).

It remains only to show that $\pi^{*!} M \otimes \mathcal{L}(-e)$ is π_* -acyclic and $\pi^{*!} M \otimes \mathcal{L}(e)$ has direct image of homological dimension 1; the other two claims follow from the facts that they are a subsheaf or quotient of a well-behaved sheaf. But

$$\mathrm{Hom}(\pi^{*!} M \otimes \mathcal{L}(-e), \mathcal{O}_e(-1)) \cong \mathrm{Hom}(\pi^{*!} M, \mathcal{O}_e(-2)) \subset \mathrm{Hom}(\pi^{*!} M, \mathcal{O}_e(-1)) = 0, \quad (6.19)$$

and similarly for

$$\mathrm{Hom}(\mathcal{O}_e(-1), \pi^{*!} M \otimes \mathcal{L}(e)). \quad (6.20)$$

□

Note that if we twist by more than just $\pm e$, then we cannot expect to obtain a well-behaved direct image. Indeed, twisting $\mathcal{O}_X \cong \pi^{*!} \mathcal{O}_Y$ by $\mathcal{L}(2e)$ gives (naturally enough) $\mathcal{L}(2e)$, which fits in an exact sequence

$$0 \rightarrow \mathcal{L}(e) \rightarrow \mathcal{L}(2e) \rightarrow \mathcal{O}_e(-2) \rightarrow 0 \quad (6.21)$$

so that $R^1 \pi_* \mathcal{O}_e(-2) \cong \mathcal{O}_p$. So we cannot iterate the twisting itself, but must instead iterate the direct-image-of-twist-of-minimal-lift operation (which we call a *pseudo-twist*). Again, we must be careful to note that the two pseudo-twists are not inverses to each other. Indeed, if M is not torsion, then the pseudo-twist by $\mathcal{L}(e)$ and the pseudo-twist by $\mathcal{L}(-e)$ change $[M] \in K_0$ by different multiples of $[\mathcal{O}_p]$, since

$$\begin{aligned} \dim \mathrm{Hom}(M, \mathcal{O}_p) - \dim \mathrm{Ext}^1(\mathcal{O}_p, M) &= \dim \mathrm{Ext}^2(\mathcal{O}_p, M) - \dim \mathrm{Ext}^1(\mathcal{O}_p, M) \\ &= \dim \mathrm{Ext}^2(\mathcal{O}_p, M) - \dim \mathrm{Ext}^1(\mathcal{O}_p, M) + \dim \mathrm{Hom}(\mathcal{O}_p, M) \\ &= \mathrm{rank}(M). \end{aligned}$$

Even if M is torsion, they need not be inverses, e.g., if M is an invertible sheaf on a curve singular at p . However, here the situation is much nicer. First, if $\pi^{*!} M$ is transverse to e , then no matter how far we twist $\pi^{*!} M$ in either direction, the result will still be a minimal lift, and thus in the transverse case, the pseudo-twists act as a group. More generally, the subsheaf $\mathcal{T}or_1(\pi^{*!} M, \mathcal{O}_e(-1)) \subset \pi^{*!} M$ will be a vector bundle on $e \cong \mathbb{P}^1$ having no global sections. If we repeatedly pseudo-twist by $\mathcal{L}(-e)$, then each step either strictly decreases the rank of this vector bundle or strictly increases its degree. Thus after a finite number of downwards pseudo-twists, we obtain a sheaf such that $\pi^{*!} M$ is transverse to the exceptional curve. We could also obtain a sheaf with transverse minimal lift via finitely many upwards pseudo-twists, or by some combination of the two; the resulting sheaves need not be in the same orbit under the group of pseudo-twists.

In generalizing pseudo-twists beyond monoidal transformations, we encounter the problem of determining which twists of a minimal lift are guaranteed to have well-behaved direct image (i.e., π_* -acyclic with direct image of homological dimension ≤ 1). This appears to be a tricky problem in general (especially in the presence of exceptional components of self-intersection < -2), but there is one sufficiently large special case.

Theorem 6.6. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, and suppose D is a divisor on X which is supported on the exceptional locus and satisfies $D^2 = -1$. If M is any sheaf of homological dimension ≤ 1 on Y , then $\pi^*M \otimes \mathcal{L}(D)$ is π_* -acyclic, with direct image of homological dimension ≤ 1 .*

Proof. Since the intersection pairing on the exceptional locus is integral and negative definite, there are at most $2n$ exceptional divisors of self-intersection -1 , where n is the number of monoidal transformations in a factorization of π . We can moreover construct those divisors easily enough. Given a factorization of π , let e_i be the total transform of the i -th exceptional curve through the remaining $n - i$ blowups; then $e_i^2 = -1$, so $\pm e_i$ satisfy the hypotheses.

In other words, we need to show that $\pi^*M \otimes \mathcal{L}(\pm e_i)$ have well-behaved direct images. Moreover, we may assume $i = n$, since twisting by $\mathcal{L}(\pm e_i)$ commutes with blowing down the last $n - i$ points. Factor $\pi = \pi_n \circ \pi_{[n]}$, where π_n is the monoidal transformation blowing down the -1 -curve e_n . Then $\pi^*M \otimes \mathcal{L}(e_n)$ is a quotient of $\pi_n^! \pi_{[n]}^* M$, so is π_* -acyclic, while $\pi^*M \otimes \mathcal{L}(-e_n)$ is a subsheaf of $\pi_n^* \pi_{[n]}^! M$, so has direct image of homological dimension ≤ 1 .

It remains to show that $\pi^*M \otimes \mathcal{L}(e_n)$ has direct image of homological dimension ≤ 1 , and $\pi^*M \otimes \mathcal{L}(-e_n)$ is π_* -acyclic. The key idea is that not only is e_n effective, but up to twisting by pullbacks of line bundles, so is $-e_n$. More precisely, on $\pi_n(X)$, we may choose a divisor D which is transverse to $\pi_{[n]}^! M$ and meets the exceptional locus only in $\pi_n(e_n)$ (this certainly exists in a formal neighborhood of the exceptional locus). Then the strict transform $\pi^*D - e_n$ of D is transverse to π^*M , so that

$$\pi^*M \otimes \mathcal{L}(e_n) \subset \pi^*(M \otimes \mathcal{L}(D)), \quad (6.22)$$

implying that $\pi_*(\pi^*M \otimes \mathcal{L}(e_n))$ has homological dimension ≤ 1 . Similarly, $\pi^*M \otimes \mathcal{L}(-e_n)$ is a quotient of $\pi^*(M \otimes \mathcal{L}(-D))$, so is π_* -acyclic. \square

Note that changing the factorization of π only permutes the divisors e_i , since they are characterized as the unique divisors of self-intersection -1 which are nonnegative linear combinations of exceptional components. Each divisor e_i is the total transform of a -1 -curve, the strict transform of which is an exceptional component f_i ; this correspondence is canonical, so that we can more naturally label the orthonormal divisors by exceptional components. I.e., for any exceptional component f , e_f is the unique effective exceptional divisor such that $e_f^2 = -1$ and $e_f \cdot f = -1$.

Corollary 6.7. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, and let f be an exceptional component of π . If M is any sheaf of homological dimension ≤ 1 on Y , then there are short exact sequences of the form*

$$0 \rightarrow M \rightarrow \pi_*(\pi^*M \otimes \mathcal{L}(e_f)) \rightarrow \mathcal{O}_{\pi(f)}^{r_1} \rightarrow 0 \quad (6.23)$$

and

$$0 \rightarrow \pi_*(\pi^*M \otimes \mathcal{L}(-e_f)) \rightarrow M \rightarrow \mathcal{O}_{\pi(f)}^{r_2} \rightarrow 0. \quad (6.24)$$

The dimensions r_1, r_2 are given by

$$r_1 = c_1(\pi^*M) \cdot e_f, \quad r_2 = c_1(\pi^*M) \cdot e_f + \text{rank}(M). \quad (6.25)$$

Proof. The first short exact sequence comes from the direct image of the sequence

$$0 \rightarrow E_1 \otimes \mathcal{L}(e_f) \rightarrow \pi^*M \otimes \mathcal{L}(e_f) \rightarrow \pi^*M \otimes \mathcal{L}(e_f) \rightarrow 0, \quad (6.26)$$

and thus $r_1 = -\chi(E_1 \otimes \mathcal{L}(e_f))$. Since $\chi(E_1) = 0$ and $\text{rank}(E_1) = 0$, Hirzebruch-Riemann-Roch gives $r_1 = -c_1(E_1) \cdot e_f$. Since

$$c_1(E_1) = \pi^* c_1(M) - c_1(\pi^{*!} M), \quad (6.27)$$

the formula for r_1 follows.

Similarly, the second short exact sequence is the direct image of

$$0 \rightarrow \pi^{*!} M \otimes \mathcal{L}(-e_f) \rightarrow \pi^! M \otimes \mathcal{L}(-e_f) \rightarrow E_2 \otimes \mathcal{L}(-e_f) \rightarrow 0, \quad (6.28)$$

so that $r_2 = \chi(E_2 \otimes \mathcal{L}(-e_f)) = -c_1(E_2) \cdot e_f$. We find

$$c_1(E_2) = c_1(\pi^! M) - c_1(\pi^{*!} M) \quad (6.29)$$

and

$$c_1(\pi^! M) = c_1(\pi^* M) + \text{rank}(M) e_\pi. \quad (6.30)$$

Since $e_\pi = \sum_f e_f$, we find $e_\pi \cdot e_f = -1$, and the formula follows. \square

Note that just as in the monoidal case, the upwards and downwards pseudo-twists do not form a group unless M is 1-dimensional and $\pi^{*!} M$ is transverse to the exceptional locus. In addition, if M is 1-dimensional and $\pi^{*!} M$ is not transverse to e_π , then we need only finitely many pseudo-twists to make it transverse. This has a particularly nice consequence in the Poisson case.

Lemma 6.8. *Let (Y, α) be a Poisson surface with anticanonical curve C_α . Let M be a pure 1-dimensional sheaf on Y (i.e., M has 1-dimensional support, and no subsheaf has 0-dimensional support) which is transverse to C_α . Then there exists a Poisson birational morphism $\pi : X \rightarrow Y$ such that some pseudo-twist M' of M has minimal lift disjoint from the anticanonical curve on X .*

Proof. If $\text{supp}(M)$ is disjoint from C_α , then there is nothing to prove. Otherwise, let $\pi_1 : Y_1 \rightarrow Y$ be the (Poisson) blow up in a point of intersection, and let M_1 be a pseudo-twist of M such that $\pi_1^{*!} M_1$ is transverse to the exceptional locus. Then $\pi_1^{*!} M_1$ is also transverse to C_{α_1} , where α_1 is the induced Poisson structure on Y_1 . Moreover,

$$c_1(\pi_1^{*!} M_1) \cdot c_1(C_{\alpha_1}) = (\pi^* c_1(M_1) - r_1 e_1) (\pi^* c_1(C_\alpha) - e_1) < c_1(M_1) \cdot c_1(C_\alpha) = c_1(M) \cdot c_1(C_\alpha). \quad (6.31)$$

(That $c_1(M_1) = c_1(M)$ is immediate from the short exact sequences for the atomic pseudo-twists.) Thus if we iterate this operation, then after at most $c_1(M) \cdot c_1(C_\alpha)$ blowups, we will achieve disjointness. \square

Remark. Since one of our motivations is to avoid using support (since this makes little sense in a noncommutative context), we should note that “pure 1-dimensional” can be rephrased strictly in terms of Hilbert polynomials: M has linear Hilbert polynomial, and no nonzero subsheaf has constant Hilbert polynomial. Similarly, the first Chern class of a pure 1-dimensional sheaf specifies how the linear term in the Hilbert polynomial depends on the choice of very ample line bundle.

If $\pi^{*!} M$ is disjoint from C_{α_X} , then we find

$$M \otimes \mathcal{O}_{C_{\alpha_Y}} \otimes \mathcal{L}(C_{\alpha_Y}) \cong \mathcal{E}xt_Y^1(\mathcal{O}_{C_{\alpha_Y}}, M) \cong \mathcal{H}om_Y(\omega_X / \pi^* \omega_Y, \pi^! M / \pi^{*!} M). \quad (6.32)$$

Moreover, disjointness has strong consequences for the invisible quotient $\pi^! M / \pi^{*!} M$. Since for any exceptional component, $f \cdot C_{\alpha_X} = f^2 + 2$, any component with $f^2 < -2$ is contained in C_{α_X} , and thus must be disjoint from $\pi^{*!} M$.

Proposition 6.9. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces. If M is a sheaf of homological dimension ≤ 1 on Y such that $\text{supp}(\pi^!M)$ does not contain any exceptional components of self-intersection ≤ -3 , then $\pi^!M/\pi^*M$ is an injective object of the category of π -invisible sheaves. More precisely,*

$$\pi^!M/\pi^*M \cong \bigoplus_f I_f^{-f \cdot c_1(\pi^!M) - \text{rank}(M)f \cdot e_\pi}. \quad (6.33)$$

Proof. Let f be any exceptional component. Since

$$\mathbf{R} \text{Hom}(\mathcal{O}_f(-1), \pi^!M) = 0, \quad (6.34)$$

we have

$$\begin{aligned} \text{Ext}^1(\mathcal{O}_f(-1), \pi^!M/\pi^*M) &\cong \text{Ext}^2(\mathcal{O}_f(-1), \pi^!M) \cong H^1(\mathcal{E}xt_X^1(\mathcal{O}_f(-1), \pi^!M)) \\ &\cong H^1(\mathcal{O}_f(f^2 + 1) \otimes \pi^*M) \cong H^0(R^1\pi_*(\mathcal{O}_f(f^2 + 1) \otimes \pi^*M)) \end{aligned} \quad (6.35)$$

If $\pi^!M$ is transverse to f , the tensor product is 0-dimensional, so acyclic. Otherwise, if $\mathcal{O}_f(f^2 + 1) \otimes \pi^*M$ (a sheaf supported on $f \cong \mathbb{P}^1$) is not acyclic, then there is a nontrivial morphism

$$\mathcal{O}_f(f^2 + 1) \otimes \pi^*M \rightarrow \mathcal{O}_f(-2), \quad (6.36)$$

so a nontrivial morphism

$$\pi^!M \rightarrow \mathcal{O}_f \otimes \pi^*M \rightarrow \mathcal{O}_f(-3 - f^2), \quad (6.37)$$

which is impossible unless $f^2 \leq -3$.

For the final claim, since $\pi^!M/\pi^*M$ is injective, it is a direct sum of injective hulls of simple sheaves, and I_f occurs as a summand with multiplicity $-f \cdot c_1(\pi^!M/\pi^*M)$, since $c_1(I_f) = c_1(P_f) = f^\vee$. The claim then follows from

$$c_1(\pi^!M/\pi^*M) = \pi^*c_1(M) + \text{rank}(M)e_\pi - c_1(\pi^!M). \quad (6.38)$$

□

Remark. Note that the hypotheses are much weaker than disjointness; in particular, if there are no exceptional components of self-intersection ≤ -3 , then $\pi^!M/\pi^*M$ is injective as an invisible sheaf for *all* sheaves M of homological dimension ≤ 1 . Conversely, if $f^2 \leq -3$ for some f , then $\pi^!\mathcal{O}_Y/\pi^*\mathcal{O}_Y$ is not injective, since

$$\text{Ext}^1(\mathcal{O}_f(-1), \pi^!\mathcal{O}_Y/\pi^*\mathcal{O}_Y) \cong \text{Ext}^2(\mathcal{O}_f(-1), \mathcal{O}_X) \cong H^0(\mathcal{O}_f(-1) \otimes \omega_X)^* \cong H^0(\mathcal{O}_f(-3 - f^2))^* \neq 0. \quad (6.39)$$

In the disjoint case, we have

$$M \otimes \mathcal{O}_{C_{\alpha_Y}} \cong M \otimes \mathcal{O}_{C_{\alpha_Y}} \otimes \mathcal{L}(C_{\alpha_Y}) \cong \bigoplus_f \text{Hom}_Y(\omega_X/\pi^*\omega_Y, I_f)^{-f \cdot c_1(\pi^!M)}. \quad (6.40)$$

Thus to complete our understanding of the disjoint case, we need to understand the 0-dimensional sheaves

$$\text{Hom}_Y(\omega_X/\pi^*\omega_Y, I_f) \cong \text{Hom}_Y(P_f, \pi^!\mathcal{O}_Y/\mathcal{O}_X). \quad (6.41)$$

Proposition 6.10. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, let f be an exceptional component, and set $d = f^\vee \cdot e_\pi$. Then there exists a homomorphism $\mathcal{O}_{Y,\pi(f)} \rightarrow k[t]/t^d$ such that*

$$\mathcal{H}om_Y(P_f, \pi^! \mathcal{O}_Y / \mathcal{O}_X) \quad (6.42)$$

is the \mathcal{O}_Y -module induced by the regular representation of $k[t]/t^d$.

Proof. Let D be a smooth divisor on X meeting the exceptional locus only in f , and that in a single reduced point p (as before, we replace X by a formal neighborhood of the exceptional locus if necessary). Then $D = \pi^* \pi_* D - f^\vee$, so that P_f fits into an exact sequence

$$0 \rightarrow P_f \rightarrow \pi^* \pi_* \mathcal{L}(D) \rightarrow \mathcal{L}(D) \rightarrow 0. \quad (6.43)$$

Since P_f is invisible, the canonical global section of $\mathcal{L}(D)$ lifts to a canonical global section of $\pi^* \pi_* \mathcal{L}(D)$, and we can thus quotient by the two copies of \mathcal{O}_X to obtain

$$0 \rightarrow P_f \rightarrow \pi^* \pi_* (\mathcal{O}_D \otimes \mathcal{L}(D)) \rightarrow \mathcal{O}_D \otimes \mathcal{L}(D) \rightarrow 0 \quad (6.44)$$

Since $\mathcal{O}_D \otimes \mathcal{L}(D)$ is transverse to the exceptional locus, it is a minimal lift, and thus we have an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{E}xt_Y^1(\mathcal{O}_D \otimes \mathcal{L}(D), \mathcal{O}_X) &\rightarrow \mathcal{E}xt_Y^1(\mathcal{O}_D \otimes \mathcal{L}(D), \pi^! \mathcal{O}_Y) \\ &\rightarrow \mathcal{H}om_Y(P_f, \pi^! \mathcal{O}_Y / \mathcal{O}_X) \rightarrow \mathcal{E}xt_Y^2(\mathcal{O}_D \otimes \mathcal{L}(D), \mathcal{O}_X) \rightarrow 0 \end{aligned}$$

Using the natural locally free resolution of $\mathcal{O}_D \otimes \mathcal{L}(D)$ turns this into a short exact sequence

$$0 \rightarrow \pi_*(\mathcal{O}_D) \rightarrow \pi_*(\mathcal{O}_D \otimes \pi^! \mathcal{O}_Y) \rightarrow \mathcal{H}om_Y(P_f, \pi^! \mathcal{O}_Y / \mathcal{O}_X) \rightarrow 0; \quad (6.45)$$

the assumptions on D mean that $R^1 \pi_* \mathcal{O}_D = 0$, and thus $\mathcal{E}xt_Y^2(\mathcal{O}_D \otimes \mathcal{L}(D), \mathcal{O}_X) = 0$. In particular, the above short exact sequence must agree with the direct image of the short exact sequence

$$0 \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_D \otimes \pi^! \mathcal{O}_Y \rightarrow J \rightarrow 0. \quad (6.46)$$

Now, since D is smooth and meets the exceptional locus in a single point, J is the structure sheaf of a jet, so has the form $k[t]/t^d$, where t is a uniformizer of D at p . The claim then follows using the composition

$$\mathcal{O}_{Y,p} \rightarrow \mathcal{O}_{D,p} \rightarrow k[t]/t^d \quad (6.47)$$

to compute the direct image. \square

Remark. Note that the homomorphism to $k[t]/t^d$ need not be surjective, reflecting the fact that $\mathcal{H}om_Y(P_f, \pi^! \mathcal{O}_Y / \mathcal{O}_X)$ need not be the structure sheaf of a jet.

The sheaf $\pi^! \mathcal{O}_Y / \mathcal{O}_X$ also has some relevant structure. If π simply blows up a collection of distinct points on Y , then any choice of ordering on those points gives a natural filtration of $\pi^! \mathcal{O}_Y / \mathcal{O}_X$. Interestingly, we have the same freedom even for much more complicated birational morphisms.

Proposition 6.11. *The invisible subsheaves of $\pi^! \mathcal{O}_Y / \mathcal{O}_X$ form a boolean lattice. The maximal chains in the lattice are in natural correspondence with the orderings on the exceptional components; if f_1, f_2, \dots, f_n is such an ordering, then there is a unique maximal chain*

$$0 = E_0 \subset E_1 \subset \dots \subset E_n = \pi^! \mathcal{O}_Y / \mathcal{O}_X \quad (6.48)$$

of invisible sheaves such that

$$c_1(E_i/E_{i-1}) = e_{f_i}. \quad (6.49)$$

Moreover, for any exceptional component f ,

$$\mathcal{H}om_Y(P_f, E_i/E_{i-1}) \cong \mathcal{O}_{\pi(f)}^{f^\vee \cdot e_{f_i}}. \quad (6.50)$$

Proof. The subsheaves of $\pi^! \mathcal{O}_Y / \mathcal{O}_X$ are in natural correspondence with the subsheaves of $\pi^! \mathcal{O}_Y$ containing \mathcal{O}_X . If M is such a subsheaf, then M/\mathcal{O}_X is invisible iff $\pi^! \mathcal{O}_Y / M$ is invisible. In particular, $\pi^! \mathcal{O}_Y / M$ cannot have any 0-dimensional subsheaves, so that it has homological dimension ≤ 1 . Thus M is locally free, and $M \cong \mathcal{L}(D)$ for some exceptional divisor D . Moreover, we compute

$$0 = \chi(M/\mathcal{O}_X) = D \cdot (e_\pi - D). \quad (6.51)$$

In particular, if we express D in terms of the orthonormal basis e_f , then every coefficient must be either 0 or 1. In other words,

$$D = \sum_{f \in S} e_f \quad (6.52)$$

for some subset S of the set of exceptional components. Since the sheaf $\mathcal{O}_D \otimes \mathcal{L}(\mathcal{O}_D)$ has Euler characteristic 0 and is contained in the invisible sheaf $\pi^! \mathcal{O}_Y / \mathcal{O}_X$, it too is invisible. We thus conclude that the invisible subsheaves of $\pi^! \mathcal{O}_Y / \mathcal{O}_X$ are in order-preserving bijection with the sets of exceptional components. Since $c_1(\mathcal{O}_D \otimes \mathcal{L}(\mathcal{O}_D)) = D$, the claim about Chern classes of subquotients in maximal chains follows.

If E is a subquotient in some maximal chain, with $c_1(E) = g$, then

$$E \cong \mathcal{O}_{e_g} \otimes \mathcal{L}\left(\sum_{h \in S} e_h\right) \quad (6.53)$$

where S is any set of exceptional components containing g . We thus need to show that for any such sheaf, $\mathcal{H}om_Y(P_f, E)$ is scheme-theoretically supported on $\mathcal{O}_{\pi(f)}$; that it has length $f^\vee \cdot e_g$ follows from a Chern class computation. Factor $\pi = \pi_1 \circ \pi_2$ with π_1 monoidal, having exceptional curve e . If $f \neq e$, then

$$P_f \cong \pi_1^* P_{\pi_1(f)}, \quad (6.54)$$

so that

$$\mathcal{H}om_Y(P_f, E) \cong \mathcal{H}om_Y(P_{\pi_1(f)}, \pi_1(E)). \quad (6.55)$$

If $g = e$, then $E \cong \mathcal{O}_g(-1)$, and thus $\pi_1(E) = 0$. Otherwise, E has the form $\pi_1^* E'$ or $\pi_1^! E'$ (depending on whether $e \in S$), where E' is a sheaf of the same form in $\pi_1(X)$. Either way, the claim follows by induction.

If $f = e$, then we may choose a curve D as above, so that

$$\mathcal{H}om_Y(P_f, E) \cong \mathcal{H}om_Y(\mathcal{O}_D \otimes \mathcal{L}(D), E) \cong \pi_*(\mathcal{O}_D \otimes E). \quad (6.56)$$

Since E is an invertible sheaf on its support, $\mathcal{H}om_Y(P_f, E)$ depends only on the support. Thus if $g \neq e$, then we may assume $E \cong \pi_1^! \pi_{1*} E'$, and thus

$$\mathcal{H}om_Y(P_f, E) \cong \mathcal{H}om_Y(\pi_{1*} P_f, \pi_{1*} E). \quad (6.57)$$

Since $\pi_{1*} P_f$ is projective, the claim follows by induction.

Finally, if $f = e = g$, then $\mathcal{H}om_Y(P_f, E)$ has length 1, so is necessarily supported on a single point. \square

We close the section with another result on resolving sheaves via minimal lifts. This does not involve pseudo-twists, but as a result requires somewhat stronger hypotheses. Here $\text{Fitt}_0(M)$ denotes the 0-th Fitting scheme of M , which for M pure 1-dimensional is a canonical divisor representing $c_1(M)$.

Proposition 6.12. *Suppose (Y, α) is a Poisson surface, and M is a pure 1-dimensional sheaf on Y such that the divisor $\text{Fitt}_0(M)$ meets C_α in a disjoint union of jets. Let $\pi : X \rightarrow Y$ be the minimal desingularization of the blowup of Y in the intersection. Then π is Poisson, and π^*M is disjoint from the induced anticanonical curve.*

Proof. We can compute π by repeatedly blowing up single points of intersection, so that π is in particular Poisson. The only issue is to show that the jet condition (and thus transversality) is preserved under the blowup $\pi : X \rightarrow Y$ in a point $p \in C_\alpha \cap \text{supp}(M)$. If C_α is smooth at p , the jet condition in a neighborhood of p is automatic, and this remains true after blowing up p .

Thus suppose C_α is singular at p . Then we have

$$M \otimes (\mathcal{O}_X/\mathcal{I}_p^2) \cong M \otimes \mathcal{O}_{C_\alpha} \otimes (\mathcal{O}_X/\mathcal{I}_p^2), \quad (6.58)$$

where \mathcal{I}_p is the ideal sheaf of p . Thus since $M \otimes \mathcal{O}_{C_\alpha}$ is a sum of jets, so is $M \otimes \mathcal{O}_X/\mathcal{I}_p^2$. But then we can compute π^*M near e using the minimal resolution of M over the local ring at p , and conclude that π^*M is transverse to the exceptional curve (meeting it in the tangent vectors to the jets through p), and thus is transverse to $\pi^*\mathcal{O}_{C_\alpha}$.

It follows that we have a short exact sequence

$$0 \rightarrow \pi_* \mathcal{E}xt_X^1(\pi^*\mathcal{O}_{C_\alpha}, \pi^*M) \rightarrow \mathcal{E}xt_Y^1(\mathcal{O}_{C_\alpha}, \pi^*M) \rightarrow \mathcal{H}om_Y(\mathcal{O}_e(-1), \pi^*M/\pi^*M) \rightarrow 0, \quad (6.59)$$

and moreover that

$$\mathcal{H}om_Y(\mathcal{O}_e(-1), \pi^*M/\pi^*M) \cong \mathcal{H}om_Y(\mathcal{E}xt_Y^1(\mathcal{O}_{C_\alpha}, \pi^*M), \mathcal{O}_p). \quad (6.60)$$

In particular,

$$\pi_* \mathcal{E}xt_X^1(\pi^*\mathcal{O}_{C_\alpha}, \pi^*M) \quad (6.61)$$

is again a sum of jets, so that the same holds for

$$\mathcal{E}xt_X^1(\pi^*\mathcal{O}_{C_\alpha}, \pi^*M). \quad (6.62)$$

□

7 Maps between Poisson moduli spaces

In this section, $\pi : X \rightarrow Y$ will be a Poisson birational morphism of Poisson surfaces, which are now over a more general base scheme S . For convenience of notation, we will silently identify sheaves on S with their pullbacks to X or Y as appropriate.

Although our goal is to show that π^* respects the Poisson structure (in a suitable sense), we will in fact mainly focus on π_* instead; since π^* is an inverse to π_* , it will be easy to derive Poissonness of π^* from Poissonness of π_* , but the latter will also let us deal with pseudo-twists.

The first issue is that direct images do not in general preserve flatness of families. This is, of course, just a relative version of semicontinuity questions, but is particularly easy to deal with in our case.

Lemma 7.1. *Let M be a coherent sheaf on X , flat over S . If $R^1\pi_*M$ is flat over S , then so is π_*M , and moreover the natural map*

$$\pi_*M \otimes N \rightarrow \pi_*(M \otimes N) \quad (7.1)$$

is an isomorphism for all coherent sheaves N on S . In particular, this holds if every fiber of M is π_ -acyclic.*

Proof. For any coherent sheaf N on S , we have an isomorphism

$$\mathbf{R}\pi_*M \otimes^{\mathbf{L}} N \cong \mathbf{R}\pi_*(M \otimes^{\mathbf{L}} N) \cong \mathbf{R}\pi_*(M \otimes N). \quad (7.2)$$

Since π has fibers of dimension ≤ 1 , the spectral sequence for $\mathbf{R}\pi_*M \otimes^{\mathbf{L}} N$ has entries in only two rows. We thus obtain isomorphisms

$$\mathcal{T}or_{p+2}(R^1\pi_*M, N) \cong \mathcal{T}or_p(\pi_*M, N) \quad (7.3)$$

for $p > 0$, an isomorphism

$$R^1\pi_*M \otimes N \cong R^1\pi_*(M \otimes N), \quad (7.4)$$

and an exact sequence

$$0 \rightarrow \mathcal{T}or_2(R^1\pi_*M, N) \rightarrow \pi_*M \otimes N \rightarrow \pi_*(M \otimes N) \rightarrow \mathcal{T}or_1(R^1\pi_*M, N) \rightarrow 0. \quad (7.5)$$

The claim follows immediately. \square

It follows that π_* induces a morphism from an open subspace of Spl_X to Spl_Y . More precisely, we have the following.

Lemma 7.2. *Let $Spl_{X,\pi} \subset Spl_X$ be the subspace of simple sheaves M which are π_* -acyclic and have simple direct image. Then $Spl_{X,\pi}$ is an open subspace, and π_* induces a surjective morphism from $Spl_{X,\pi}$ to Spl_Y .*

Proof. That π_* -acyclicity is an open condition follows from the fact that for any S -flat family M , the functor $R^1\pi_*$ commutes with taking fibers; thus the π_* -acyclic locus is just the complement of the image of the closed subscheme $\text{Fitt}_0(R^1\pi_*M)$ under the proper map $X \rightarrow S$. Then π_*M is a flat family, and simplicity is an open condition on flat families [1].

For surjectivity, note that point sheaves in Spl_Y are direct images of point sheaves in Spl_X , while any other simple sheaf is the direct image of its minimal lift. \square

We will need one more fact in the proof of Poissonness, which we will also use in the next section, so separate out from the proof.

Lemma 7.3. *Let M be a π_* -acyclic sheaf on X with direct image of homological dimension ≤ 1 . Then there is a commutative diagram*

$$\begin{array}{ccc} \pi^!\pi_*M \otimes \pi^*\omega_Y & \xrightarrow{1 \otimes \pi^*\alpha_Y} & \pi^!\pi_*M \\ \downarrow & & \uparrow \\ M \otimes \omega_X & \xrightarrow{1 \otimes \alpha_X} & M, \end{array} \quad (7.6)$$

where the first vertical map factors as

$$\pi^!\pi_*M \otimes \pi^*\omega_Y \rightarrow \pi^*\pi_*M \otimes \omega_X \rightarrow M \otimes \omega_X. \quad (7.7)$$

Proof. Since π is an isomorphism outside the exceptional locus, the diagram can only fail to commute on the exceptional locus. The failure to commute is thus measured by a morphism

$$\pi^! \pi_* M \otimes \pi^* \omega_Y \rightarrow \pi^! \pi_* M \quad (7.8)$$

that vanishes outside the exceptional locus. We can thus apply Proposition 5.4 to conclude that this morphism has invisible image. Since $\pi^! \pi_* M$ has no invisible subsheaf, the image is 0, and thus the diagram commutes. \square

Theorem 7.4. *Let $\pi : X \rightarrow Y$ be a Poisson birational morphism of Poisson surfaces. The direct image functor π_* defines a Poisson morphism $\pi_* : \mathcal{S}pl_{X,\pi} \rightarrow \mathcal{S}pl_Y$.*

Proof. We have already shown that it defines a morphism, so it remains only to show that it respects the Poisson structure. Consider a sheaf M in $\mathcal{S}pl_{X,\pi}$. If $\pi_* M$ has homological dimension 2, then $\pi_* M$ must be a point, and either M is a point or M is supported on the exceptional locus. On the subspace $X \subset \mathcal{S}pl_X$ parametrizing point sheaves, π_* is just π , so is certainly Poisson. If M is supported on the exceptional locus, then $\pi_* M$ is supported on the anticanonical curve, so that for such sheaves, π_* maps to a single point with trivial Poisson structure, so again is Poisson.

We may thus restrict our attention to the case that $\pi_* M$ has homological dimension ≤ 1 . Now, the adjunction between $\pi^!$ and π_* induces natural maps

$$\begin{aligned} \mathrm{Ext}^1(\pi_* M, \pi_* M \otimes \omega_Y) &\cong \mathrm{Ext}^1(M, \pi^! \pi_* M \otimes \pi^* \omega_Y) \\ \mathrm{Ext}^1(\pi_* M, \pi_* M) &\cong \mathrm{Ext}^1(M, \pi^! \pi_* M), \end{aligned}$$

with inverse given by π_* . (To be precise, given a π_* -acyclic sheaf N , there is a natural map $\pi_* : \mathrm{Ext}^1(M, N) \rightarrow \mathrm{Ext}^1(\pi_* M, \pi_* N)$ given by taking the direct image of the extension.) In particular, the Poisson structure on a neighborhood of $\pi_* M$ in $\mathcal{S}pl_Y$ is obtained by composing the above isomorphisms with the map

$$\mathrm{Ext}^1(M, \pi^! \pi_* M \otimes \pi^* \omega_Y) \xrightarrow{\mathrm{Ext}^1(M, 1 \otimes \pi^* \alpha_Y)} \mathrm{Ext}^1(M, \pi^! \pi_* M). \quad (7.9)$$

By Lemma 7.3, this factors through the natural map

$$\mathrm{Ext}^1(M, M \otimes \omega_X) \xrightarrow{\mathrm{Ext}^1(M, 1 \otimes \alpha_X)} \mathrm{Ext}^1(M, M), \quad (7.10)$$

and the composition

$$\mathrm{Ext}^1(M, M) \rightarrow \mathrm{Ext}^1(M, \pi^! \pi_* M) \cong \mathrm{Ext}^1(\pi_* M, \pi_* M) \quad (7.11)$$

is just π_* , i.e., the differential of the morphism $\pi_* : \mathcal{S}pl_{X,\pi} \rightarrow \mathcal{S}pl_Y$. In other words, the map $\Omega_{\mathcal{S}pl_Y} \rightarrow \Omega_{\mathcal{S}pl_Y}^*$ induced by the Poisson structure is the direct image of the corresponding map on $\mathcal{S}pl_{X,\pi}$, and thus π_* is Poisson. \square

Corollary 7.5. *The functors π^* and $\pi^!$ induce Poisson morphisms from $\mathcal{S}pl_Y$ to $\mathcal{S}pl_{X,\pi}$.*

Proof. Indeed, π^* and $\pi^!$ are injective, and preserve simplicity and flatness. Since the images of π^* and $\pi^!$ satisfy the hypotheses of the Theorem, π_* is Poisson on both images; since it is also an isomorphism on both images, the inverses are Poisson. But this is precisely what we want to prove. \square

For the minimal lift, we again have an issue with flatness. In this case, however, we can completely control the corresponding flattening stratification. Let $Spl_{Y, \leq 1}$ denote the subspace parametrizing sheaves of homological dimension ≤ 1 .

Lemma 7.6. *Suppose π is monoidal, blowing up the point $p \in Y$. Then for each integer $m \geq 0$, the subspace of $Spl_{Y, \leq 1}$ parametrizing sheaves M with $\dim(\text{Ext}^1(\mathcal{O}_p, M)) = m$ is a Poisson subspace, and contains every symplectic leaf it intersects.*

Proof. Since π is Poisson, p is contained in the anticanonical curve C_α , and thus

$$M \otimes \mathcal{O}_p \cong (M \otimes \mathcal{O}_{C_\alpha}) \otimes \mathcal{O}_p \quad (7.12)$$

is constant on symplectic leaves, so determines a stratification by locally closed Poisson subspaces. Since

$$\dim(H^0(M \otimes \mathcal{O}_p)) = \dim(\text{Hom}(M, \mathcal{O}_p)) = \dim(\text{Ext}^1(\mathcal{O}_p, M)) + \text{rank}(M), \quad (7.13)$$

the same applies to $\dim(\text{Ext}^1(\mathcal{O}_p, M))$. \square

Now, given any flat family of sheaves on Y with homological dimension ≤ 1 , we obtain a stratification of the base of the family by taking the flattening stratification of the minimal lift of the family. Normally this depends on a choice of relatively very ample bundle, but one can always refine such a stratification by imposing the open and closed conditions that the numerical Chern classes be constant, and the result will then be independent of the relatively very ample bundle. It is this canonical stratification that we will mean; it refines the usual decomposition of Y by numerical Chern class.

Corollary 7.7. *If π is monoidal, then the stratification induced on $Spl_{Y, \leq 1}$ by π agrees with the stratification by $\dim \text{Ext}^1(\mathcal{O}_p, M)$ and numerical Chern class.*

Proof. Indeed, for any coherent sheaf M of homological dimension ≤ 1 , we have the equation

$$[\pi^{*!}M] = \pi^*[M] - \dim \text{Ext}^1(\mathcal{O}_p, M)[\mathcal{O}_e(-1)] \quad (7.14)$$

in the Grothendieck group. It follows that $\pi^{*!}M$ has constant numerical Chern class iff M has constant numerical Chern class and $\dim \text{Ext}^1(\mathcal{O}_p, M)$ is constant. \square

Theorem 7.8. *Let $\pi : X \rightarrow Y$ be a Poisson birational morphism of Poisson surfaces. Every stratum of the stratification induced on $Spl_{Y, \leq 1}$ by π is a Poisson subspace, and the minimal lift induces a Poisson isomorphism from each stratum to an open subspace of $Spl_{X, \leq 1}$.*

Proof. First suppose that π is monoidal. Then we have already shown that the stratification is Poisson. On each stratum, the minimal lift preserves flatness (and simplicity), so defines a morphism from the stratum to $Spl_{X, \leq 1}$. That the image is open follows from semicontinuity and the fact that minimal lifts are characterized by the vanishing of $\text{Hom}(\mathcal{O}_e(-1), M)$ and $\text{Hom}(M, \mathcal{O}_e(-1))$.

We thus obtain an isomorphism between each stratum and the corresponding open subspace, and it remains only to show that it is Poisson. Equivalently, we need to show that π_* is Poisson on the image of $\pi^{*!}$, but this is immediate from Theorem 7.4.

More generally, if we factor $\pi = \pi_1 \circ \pi_2$ with π_1 monoidal, then the stratification induced by π refines the stratification induced by π_2 , and is identified via $\pi_2^{*!}$ with the stratification induced by π_1 on $\pi_1(X)$. The claim then follows easily by induction. \square

Corollary 7.9. *Let $\pi : X \rightarrow Y$ be a birational morphism of Poisson surfaces. Then for every stratum of the stratification induced by π on $\mathcal{S}pl_{Y, \leq 1}$, any pseudo-twist operation defines a Poisson morphism on the open subspace of the stratum where the pseudo-twist is simple.*

Proof. Indeed, a pseudo-twist is a composition of the Poisson morphism $\pi^{*!}$, the Poisson morphism $-\otimes \mathcal{L}(\pm e_f)$, and the (partially defined) Poisson morphism π_* . \square

Corollary 7.10. *Let (Y, α) be a Poisson surface, let the subscheme $J \subset C_\alpha$ be a disjoint union of jets, and let $\pi : X \rightarrow Y$ be the minimal desingularization of the blowup of Y along J . For any sheaf M_α which is a direct sum of structure sheaves of subschemes of J , $\pi^{*!}$ is a symplectomorphism on the symplectic leaf $M \otimes \mathcal{O}_{C_\alpha} = M_\alpha$ of $\mathcal{S}pl_{Y, \leq 1}$.*

Proof. Indeed, $M \otimes \mathcal{O}_{C_\alpha} = M_\alpha$ iff $\pi^{*!}M$ is disjoint from C_α , and has the correct numerical Chern class. \square

Remark. In particular, if C_α is smooth, then any symplectic leaf consisting of sheaves transverse to C_α is symplectomorphic to an open subspace of some $\mathcal{S}pl_X$ with $\pi : X \rightarrow Y$ a Poisson birational morphism: simply choose $\pi : X \rightarrow Y$ so that $\pi^{*!}M$ is disjoint from C_α .

The fact that inverse images and minimal lifts behave nicely with respect to the Poisson structure has an interesting consequence. Let $f : X \dashrightarrow Y$ be a Poisson birational map. Then we can use the above facts to essentially embed $\mathcal{S}pl_X$ as a Poisson subspace of $\mathcal{V}ect_Y$. More precisely, suppose $U \rightarrow \mathcal{S}pl_{X, \leq 1}$ is a Noetherian étale neighborhood (in the remaining case, when U is an étale neighborhood of X , simply consider the corresponding family of ideal sheaves). Then there is a corresponding morphism $U \rightarrow \mathcal{V}ect_Y$ (étale to its image) such that the two induced Poisson structures are the same.

To see this, let Z be a Poisson resolution of f , so that f factors through Poisson birational morphisms $g : Z \rightarrow X$ and $h : Z \rightarrow Y$. Then if M is the original flat family of sheaves on X , g^*M is a flat family of sheaves on Z , with the same Poisson structure. We may then apply the construction of [8] (see Section 3.2 above) to turn this into a flat family of simple locally free sheaves on Z (negating the Poisson structure in the process). Let V be this family. Now, if we twist V by a suitable line bundle, we may arrange to have

$$\mathrm{Hom}(V, \mathcal{O}_e(-1)) = 0 \tag{7.15}$$

for every h -exceptional component e . Indeed, we have

$$\mathrm{Hom}(V, \mathcal{O}_e(-1)) \cong \mathrm{Hom}(V|_e, \mathcal{O}_e(-1)) \cong H^1(V|_e(-1))^* \tag{7.16}$$

so we need simply twist by a relatively ample bundle that makes $\bigoplus_e V|_e(-1)$ acyclic. But this makes V a minimal lift under $h!$. Indeed, since V is torsion-free, we also have $\mathrm{Hom}(\mathcal{O}_e(-1), V) = 0$ for all e , and thus Theorem 5.15 applies. In particular, it follows that h_*V is simple, and induces an anti-Poisson map from U to $\mathcal{S}pl_{Y, \leq 1}$. Applying the construction of [8] again gives the required Poisson morphism $U \rightarrow \mathcal{V}ect_Y$. Each step of the above process is reversible, so the maps $U \rightarrow \mathcal{S}pl_{X, \leq 1}$ and $U \rightarrow \mathcal{V}ect_Y$ have isomorphic images (algebraic subspaces of $\mathcal{S}pl_{X, \leq 1}$, $\mathcal{V}ect_Y$ respectively).

One possible application of this construction is in the noncommutative context. The discussion of Section 3.2 essentially never uses the fact that X is a *commutative* surface; the main change needed (assuming X reasonably well-behaved) is to replace the functor $-\otimes \omega_X$ by the appropriate analogue for noncommutative Serre duality.¹ The only other significant change is that V is no

¹There are also some minor technicalities involving the fact that twisting by a line bundle changes the noncommutative surface; of course, one can always twist back.

longer locally free in any obvious sense, but we can still hope to show that some twist of V has well-behaved direct image (i.e., V is h_* -acyclic, with simple image from which we can reconstruct V), in which case the above construction can still be carried out.² The construction would then give us the ability to finesse a difficulty of the noncommutative context, namely the fact that it is unclear how to define “locally free” on a general noncommutative surface, and thus even less clear how to prove that the corresponding moduli space is Poisson. This construction would mean it sufficed to show the Poisson property for only one surface from each birational equivalence class (e.g., for noncommutative ruled surfaces). In particular, the case of noncommutative rational surfaces would reduce to that of noncommutative \mathbb{P}^2 , in which case one may use more ad-hoc calculations [14] to prove the Jacobi identity and classify the symplectic leaves.

In addition to images and lifts, there is one more natural morphism we want to consider. We have already discussed the dualization functor

$$-^D := \mathcal{E}xt_Y^1(-, \omega_Y) \quad (7.17)$$

in the context of invisible sheaves, but of course much of what we have said applies to any pure 1-dimensional sheaf. Duality also interacts nicely with the inverse image and minimal lift functors.

Proposition 7.11. *Let $\pi : X \rightarrow Y$ be a birational morphism of smooth projective surfaces. Then for any pure 1-dimensional sheaf M on Y , π^*M , $\pi^!M$, and $\pi^{*!}M$ are pure 1-dimensional sheaves, and we have*

$$\begin{aligned} (\pi^*M)^D &\cong \pi^!(M^D) \\ (\pi^{*!}M)^D &\cong \pi^{*!}(M^D) \\ (\pi^!M)^D &\cong \pi^*(M^D). \end{aligned}$$

Proof. This is essentially by inspection using a locally free resolution

$$0 \rightarrow V \rightarrow W \rightarrow M \rightarrow 0 \quad (7.18)$$

of M , and the corresponding resolution

$$0 \rightarrow W^* \rightarrow V^* \rightarrow M^D \rightarrow 0 \quad (7.19)$$

of M^D . □

Since duality interacts nicely with these Poisson morphisms, it is natural to suspect that it is itself Poisson. This is not quite the case: it is in fact anti-Poisson.

Proposition 7.12. *Let (X, α) be a Poisson surface, and let $\mathcal{S}pl_X^1$ be the subspace parametrizing pure 1-dimensional sheaves. Then $-^D$ is an anti-Poisson involution on $\mathcal{S}pl_X^1$.*

Proof. Recall that at a point $M \in \mathcal{S}pl_X^1$, the pairing on the sheaf of differentials is given by the composition

$$\mathrm{Ext}^1(M, M \otimes \omega_X) \otimes \mathrm{Ext}^1(M, M \otimes \omega_X) \rightarrow \mathrm{Ext}^2(M, M \otimes \omega_X^2) \rightarrow \mathrm{Ext}^2(M, M \otimes \omega_X) \rightarrow k. \quad (7.20)$$

Now, $-^D$ induces an isomorphism

$$\mathrm{Ext}^1(M, M \otimes \omega_X) \rightarrow \mathrm{Ext}^1(M^D \otimes \omega_X^{-1}, M^D) \rightarrow \mathrm{Ext}^1(M^D, M^D \otimes \omega_X). \quad (7.21)$$

²Of course, it would be enough to consider the case that f is monoidal or inverse-monoidal.

Indeed, a class in $\text{Ext}^1(M, M \otimes \omega_X)$ corresponds to an extension

$$0 \rightarrow M \otimes \omega_X \rightarrow N \rightarrow M \rightarrow 0. \quad (7.22)$$

Since N is an extension of pure 1-dimensional sheaves, it is also pure 1-dimensional, and thus we can dualize the entire exact sequence to obtain

$$0 \rightarrow M^D \rightarrow N^D \rightarrow (M \otimes \omega_X)^D \rightarrow 0, \quad (7.23)$$

and note that $(M \otimes \omega_X)^D \cong M^D \otimes \omega_X^{-1}$.

Similarly, the product of two such extensions is given by a four-term exact sequence of pure 1-dimensional sheaves, which we can again dualize. If ϕ, ψ are two classes in $\text{Ext}^1(M, M \otimes \omega_X)$, then we have

$$(\phi\psi)^D = \psi^D \phi^D = -\phi^D \psi^D. \quad (7.24)$$

Duality respects the trace map, so all told gives an anti-Poisson involution. \square

8 Rigidity

In studying moduli spaces of differential and difference equations, one particularly interesting question is when those moduli spaces consist of a single point. For instance, the hypergeometric differential equation has been known since Riemann to be the unique second-order Fuchsian equation with three singular points (and any specific equation is determined by the exponents at the singularities). In a generalized Hitchin system context, the order and singularity structure precisely specifies a symplectic leaf in the corresponding Poisson moduli space. We are thus interested in understanding which sheaves are rigid, i.e., the only point of their symplectic leaf. More precisely, we say that a sheaf M of homological dimension ≤ 1 is rigid if it is isomorphic to any sheaf of homological dimension ≤ 1 with the same numerical Chern class and (derived) restriction to C_α . (This is stronger than the symplectic leaf condition, since the symplectic leaf by definition includes only simple sheaves.)

A slightly easier question is infinitesimal rigidity: at which sheaves is the tangent space to the symplectic leaf 0-dimensional? Of course, a sheaf is infinitesimally rigid iff the map

$$\text{Ext}^1(M, M \otimes \omega_X) \rightarrow \text{Ext}^1(M, M) \quad (8.1)$$

is 0, or equivalently if the image is 0-dimensional. As usual, it will be convenient to embed this in an Euler characteristic; we also extend to pairs of sheaves. Thus for two coherent sheaves M, N on the Poisson surface (X, α) , we define

$$\chi_\alpha(M, N) := \dim(\text{Hom}(M, N)) - \dim(\text{im}(\text{Ext}^1(M, N \otimes \omega_X) \rightarrow \text{Ext}^1(M, N))) + \dim(\text{Hom}(N, M)). \quad (8.2)$$

Note that duality immediately gives $\chi_\alpha(M, N) = \chi_\alpha(N, M)$; it also gives

$$\dim(\text{Hom}(N, M)) = \dim(\text{Ext}^2(M, N \otimes \omega_X)), \quad (8.3)$$

making χ_α look more like an Euler characteristic.

Proposition 8.1. *The α -twisted Euler characteristic $\chi_\alpha(M, N)$ depends only on the numerical Chern classes $c_*(M)$, $c_*(N)$ and the derived restrictions $M|_{C_\alpha}^{\mathbf{L}}$ and $N|_{C_\alpha}^{\mathbf{L}}$.*

Proof. We first note that we have a long exact sequence

$$0 \rightarrow C^{-1} \rightarrow A^0 \rightarrow B^0 \rightarrow C^0 \rightarrow A^1 \rightarrow B^1 \rightarrow C^1 \rightarrow A^2 \rightarrow B^2 \rightarrow C^2 \rightarrow 0 \quad (8.4)$$

where

$$\begin{aligned} A^p &:= \operatorname{Ext}^p(M, N \otimes \omega_X), \\ B^p &:= \operatorname{Ext}^p(M, N), \\ C^p &:= \operatorname{Ext}^p(M|_{C_\alpha}^{\mathbf{L}}, N|_{C_\alpha}^{\mathbf{L}}). \end{aligned}$$

Indeed, since the complex $N \otimes \omega_X \rightarrow N$ represents $N \otimes^{\mathbf{L}} \mathcal{O}_{C_\alpha}$, to construct such a sequence, it will suffice to give a quasi-isomorphism

$$\mathbf{R}\operatorname{Hom}(M, N \otimes^{\mathbf{L}} \mathcal{O}_{C_\alpha}) \cong \mathbf{R}\operatorname{Hom}(M|_{C_\alpha}^{\mathbf{L}}, N|_{C_\alpha}^{\mathbf{L}}). \quad (8.5)$$

If $i : C_\alpha \rightarrow X$ is the natural closed embedding, then

$$N \otimes^{\mathbf{L}} \mathcal{O}_{C_\alpha} \cong \mathbf{R}i_* \mathbf{L}i^* N, \quad (8.6)$$

and thus

$$\mathbf{R}\operatorname{Hom}(M, N \otimes^{\mathbf{L}} \mathcal{O}_{C_\alpha}) \cong \mathbf{R}\operatorname{Hom}(M, \mathbf{R}i_* \mathbf{L}i^* N) \cong \mathbf{R}\operatorname{Hom}(\mathbf{L}i^* M, \mathbf{L}i^* N), \quad (8.7)$$

Since $\mathbf{L}i^* M \cong M|_{C_\alpha}^{\mathbf{L}}$, the claim follows.

Then, by definition,

$$\chi_\alpha(M, N) = \dim(B^0) - \dim(\operatorname{im}(A^1 \rightarrow B^1)) + \dim(A^2), \quad (8.8)$$

and exactness gives

$$\dim(\operatorname{im}(A^1 \rightarrow B^1)) = \dim(A^1) - \dim(C^0) + \dim(B^0) - \dim(A^0) + \dim(C^{-1}), \quad (8.9)$$

so that

$$\chi_\alpha(M, N) = \dim(A^0) - \dim(A^1) + \dim(A^2) + \dim(C^0) - \dim(C^{-1}). \quad (8.10)$$

The C dimensions manifestly only depend on $M|_{C_\alpha}^{\mathbf{L}}$ and $N|_{C_\alpha}^{\mathbf{L}}$, while

$$\dim(A^0) - \dim(A^1) + \dim(A^2) = \dim \operatorname{Ext}^2(N, M) - \dim \operatorname{Ext}^1(N, M) + \dim \operatorname{Hom}(N, M) \quad (8.11)$$

is the usual Ext Euler characteristic, and thus only depends on the numerical Chern classes of M and N (and can be computed via Hirzebruch-Riemann-Roch). \square

Remark. For convenient reference, we note that

$$\begin{aligned} &\dim \operatorname{Ext}^2(N, M) - \dim \operatorname{Ext}^1(N, M) + \dim \operatorname{Hom}(N, M) \\ &= -\chi(\mathcal{O}_X) \operatorname{rank}(M) \operatorname{rank}(N) + \chi(M) \operatorname{rank}(N) + \operatorname{rank}(M) \chi(N) \\ &\quad - c_1(M) \cdot c_1(N) + \operatorname{rank}(M) K_X \cdot c_1(N). \end{aligned} \quad (8.12)$$

If $\mathcal{T}or_1(M, \mathcal{O}_{C_\alpha}) = \mathcal{T}or_1(N, \mathcal{O}_{C_\alpha}) = 0$, then $\chi_\alpha(M, N)$ is given by the above together with the contribution $\dim \operatorname{Hom}(M|_{C_\alpha}^{\mathbf{L}}, N|_{C_\alpha}^{\mathbf{L}})$ from the restriction to the anticanonical curve. Particularly nice is the case $\operatorname{rank}(M) = \operatorname{rank}(N) = 0$, in which case

$$\chi_\alpha(M, N) = -c_1(M) \cdot c_1(N) + \dim \operatorname{Ext}^0(M|_{C_\alpha}^{\mathbf{L}}, N|_{C_\alpha}^{\mathbf{L}}) - \dim \operatorname{Ext}^{-1}(M|_{C_\alpha}^{\mathbf{L}}, N|_{C_\alpha}^{\mathbf{L}}). \quad (8.13)$$

In the case $M = N$, we call $\chi_\alpha(M, M)$ the *index of rigidity* of M , following [12]; note

$$\chi_\alpha(M, M) = 2 \dim(\text{End}(M)) - \dim(\text{im}(\text{Ext}^1(M, M \otimes \omega_X) \rightarrow \text{Ext}^1(M, M))). \quad (8.14)$$

In particular, if M is simple, then it is infinitesimally rigid iff $\chi_\alpha(M, M) = 2$.

The α -twisted Euler characteristic, and thus the index of rigidity, behaves nicely under direct images and minimal lifts. Actually, the middle term (which is the one of greatest interest in any case) has the weakest hypotheses to ensure good behavior.

Proposition 8.2. *Let $\pi : X \rightarrow Y$ be a Poisson birational morphism of Poisson surfaces, and suppose M and N are coherent sheaves on X , at least one of which is π_* -acyclic with direct image of homological dimension ≤ 1 . Then*

$$\dim(\text{im}(\text{Ext}^1(\pi_*M, \pi_*N \otimes \omega_Y) \rightarrow \text{Ext}^1(\pi_*M, \pi_*N))) \leq \dim(\text{im}(\text{Ext}^1(M, N \otimes \omega_X) \rightarrow \text{Ext}^1(M, N))). \quad (8.15)$$

Proof. By symmetry, we may assume that N is π_* -acyclic with direct image of homological dimension ≤ 1 . Then, using adjointness, we observe that the maps

$$\text{Ext}^1(\pi_*M, \pi_*N \otimes \omega_Y) \rightarrow \text{Ext}^1(\pi_*M, \pi_*N) \quad (8.16)$$

and

$$\text{Ext}^1(M, \pi^! \pi_*N \otimes \pi^* \omega_Y) \rightarrow \text{Ext}^1(M, \pi^! \pi_*N) \quad (8.17)$$

have isomorphic images. By Lemma 7.3, the latter factors through the map

$$\text{Ext}^1(M, N \otimes \omega_X) \rightarrow \text{Ext}^1(M, N), \quad (8.18)$$

giving us a bound on the image. \square

Proposition 8.3. *Let $\pi : X \rightarrow Y$ be a birational morphism of algebraic surfaces, and suppose M and N are coherent sheaves on X such that M is π_* -acyclic, π_*N has homological dimension 1, and either M has no invisible quotient or N has no invisible subsheaf. Then $\dim \text{Hom}(\pi_*M, \pi_*N) \geq \dim(\text{Hom}(M, N))$.*

Proof. It suffices to show that the natural map $\text{Hom}(M, N) \rightarrow \text{Hom}(\pi_*M, \pi_*N)$ is injective. Suppose $f : M \rightarrow N$ is in the kernel. Then it is 0 outside the invisible locus, so has invisible image, giving both an invisible quotient of M and an invisible subsheaf of N . \square

Combining hypotheses gives the following.

Corollary 8.4. *Let M, N be coherent π_* -acyclic sheaves on X with direct images of homological dimension ≤ 1 . If (1) M has no invisible quotient or N has no invisible subsheaf, and (2) N has no invisible quotient or M has no invisible subsheaf, then $\chi_\alpha(M, N) \leq \chi_\alpha(\pi_*M, \pi_*N)$.*

Corollary 8.5. *Let M be a coherent π_* -acyclic sheaf on X with direct image of homological dimension ≤ 1 . Suppose further that either M is simple, M has no invisible quotient, or M has no invisible subsheaf. Then $\chi_\alpha(M, M) \leq \chi_\alpha(\pi_*M, \pi_*M)$.*

Proof. If M has no invisible quotient or no invisible subsheaf, this is just a special case of the proposition. Those hypotheses were only used to prove the inequalities on Hom spaces, which are automatic when M is simple. \square

Proposition 8.6. *If M, N are coherent sheaves on Y of homological dimension ≤ 1 , then the minimal lift respects the α -twisted Euler characteristic: $\chi_\alpha(\pi^{*!}M, \pi^{*!}N) = \chi_\alpha(M, N)$.*

Proof. From the remark following Lemma 6.1, we know that

$$\mathrm{Hom}(\pi^{*!}M, \pi^{*!}N) \cong \mathrm{Hom}(M, N) \quad (8.19)$$

and

$$\mathrm{Ext}^1(\pi^{*!}M, \pi^{*!}N) \subset \mathrm{Ext}^1(M, N). \quad (8.20)$$

Dually, the map

$$\mathrm{Ext}^1(M, N \otimes \omega_Y) \rightarrow \mathrm{Ext}^1(\pi^{*!}M, \pi^{*!}N \otimes \omega_X) \quad (8.21)$$

is surjective. But then the maps

$$\mathrm{Ext}^1(\pi^{*!}M, \pi^{*!}N \otimes \omega_X) \rightarrow \mathrm{Ext}^1(\pi^{*!}M, \pi^{*!}N) \quad (8.22)$$

and

$$\mathrm{Ext}^1(M, N \otimes \omega_Y) \rightarrow \mathrm{Ext}^1(M, N) \quad (8.23)$$

have isomorphic images. \square

Proposition 8.7. *Let $\pi : X \rightarrow Y$ be a birational morphism of Poisson surfaces, and M a coherent sheaf on Y of homological dimension ≤ 1 . If M' is any pseudo-twist of M , then $\chi_\alpha(M, M) \leq \chi_\alpha(M', M')$.*

Proof. Since $\chi_\alpha(M, M) = \chi(\pi^{*!}M, \pi^{*!}M) = \chi_\alpha(\pi^{*!}M \otimes \mathcal{L}(\pm e_f), \pi^{*!}M \otimes \mathcal{L}(\pm e_f))$, it remains to show that the last index of rigidity is nonincreasing under direct images. That this holds for the Ext^1 term follows from the fact that $\pi^{*!}M \otimes \mathcal{L}(\pm e_f)$ is π_* -acyclic with direct image of homological dimension ≤ 1 . For the $2 \dim(\mathrm{End}(M))$ term, consider the composition

$$\mathrm{End}(M) \cong \mathrm{End}(\pi^{*!}M) \cong \mathrm{End}(\pi^{*!}M \otimes \mathcal{L}(\pm e_f)) \rightarrow \mathrm{End}(M'). \quad (8.24)$$

This is an isomorphism away from the point $\pi(f)$, and thus any endomorphism in the kernel must vanish away from this point. In particular, any such endomorphism would have 0-dimensional image, which must be 0 since M has homological dimension ≤ 1 . \square

In the Hitchin system context, the main question is which pure 1-dimensional sheaves are rigid. A first step is the following.

Lemma 8.8. *If M is a simple 1-dimensional sheaf on the Poisson surface Y , transverse to the anticanonical curve, then M is rigid iff it is infinitesimally rigid and $\mathrm{Fitt}_0(M)$ is an integral curve.*

Proof. If M has integral support and is infinitesimally rigid, then since it is simple, we have $\chi_\alpha(M, M) = 2$. It follows that for any other sheaf N of homological dimension ≤ 1 with the same numerical Chern class and restriction to C_α , we have

$$\chi_\alpha(M, N) = 2. \quad (8.25)$$

But this implies either $\mathrm{Hom}(M, N) > 0$ or $\mathrm{Hom}(N, M) > 0$. Since M is integral, any subsheaf is either 0 or has 0-dimensional quotient. Thus if $f : M \rightarrow N$ is a nonzero morphism, then either f is injective, or f has 0-dimensional image. The latter cannot happen, since N has homological dimension ≤ 1 , and the former implies that f is an isomorphism, by comparison of Chern classes.

Similarly, if $f : N \rightarrow M$ is a nonzero morphism, then the cokernel is 0-dimensional, but then comparison of Chern classes shows that the kernel must also be 0-dimensional, again a contradiction unless f is an isomorphism. Either way, we obtain an isomorphism between M and N , and thus M is rigid.

Conversely, if M is rigid, then it is certainly infinitesimally rigid (since symplectic leaves in $\mathcal{S}pl_Y$ are smooth), so only the support needs to be controlled. Twisting by a line bundle has no effect on rigidity, so we may assume that M has no global sections. If $\text{Fitt}_0(M)$ is not integral, then we may choose a nonzero subsheaf N with strictly smaller support, and a point p of the divisor $\text{Fitt}_0(M) - \text{Fitt}_0(N)$ which is not on C_α . For some sufficiently ample divisor D on Y , the twist $N' = N \otimes \mathcal{L}(D)$ will have a global section. Define a sequence of sheaves

$$M \otimes \mathcal{L}(D) \cong M_{D \cdot c_1(M)} \supset M_{D \cdot c_1(M) - 1} \supset \cdots \supset M_0, \quad (8.26)$$

all containing N' , in the following way. To obtain M_{k-1} from M_k , choose a nonzero map $M_k/N' \rightarrow \mathcal{O}_p$; such a map exists since p is in the support of M_k/N' . Then M_{k-1} is the kernel of the induced map $M_k \rightarrow \mathcal{O}_p$.

This process has no effect on the first Chern class, and each step reduces the Euler characteristic by 1, so that M_0 has the same Chern class as M . In addition, twisting by $\mathcal{L}(D)$ has no effect on the restriction to C_α (by transversality), and each inclusion in the sequence is an isomorphism away from p . So rigidity implies that $M \cong M_0$. But M has no global sections, while $M_0 \supset N'$ does. \square

Remark. Note that the argument that a map from M to N must be an isomorphism also shows that any pure 1-dimensional sheaf with integral Fitting scheme is simple. The argument from constancy of the index of rigidity is essentially that of [12, Thm. 1.1.2].

In many cases, the above argument ruling out non-integral support can be interpreted as twisting by a suitable invertible sheaf on the support, and noting that the resulting sheaf is nonisomorphic. We could try to do something similar in cases with integral support and positive genus; the main technicality is that the action of invertible sheaves on torsion-free sheaves has nontrivial stabilizers. However, if we attempt to identify the part of $\text{Ext}^1(M, M)$ coming from such twisting, we are led to the following result.

Lemma 8.9. *Let M be a pure 1-dimensional sheaf on X transverse to the anticanonical curve. Then there is a natural injection*

$$H^1(\mathcal{H}om(M, M)) \rightarrow \text{im}(\text{Ext}^1(M, M \otimes \omega_X) \rightarrow \text{Ext}^1(M, M)). \quad (8.27)$$

Proof. The local-global spectral sequence for $\text{Ext}^*(M, M)$ collapses at the E_2 page; indeed, this happens for $\text{Ext}^*(M, N)$ as long as M has homological dimension 1 and N has ≤ 1 -dimensional support. We thus obtain the following short exact sequence

$$0 \rightarrow H^1(\mathcal{H}om(M, M)) \rightarrow \text{Ext}^1(M, M) \rightarrow H^0(\mathcal{E}xt^1(M, M)) \rightarrow 0. \quad (8.28)$$

Similarly, the spectral sequence for $\text{Ext}^*(M, M \otimes \omega_X)$ collapses, giving a corresponding exact sequence and a commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{H}om(M, M) \otimes \omega_X) & \longrightarrow & \text{Ext}^1(M, M \otimes \omega_X) \\ \downarrow & & \downarrow \\ H^1(\mathcal{H}om(M, M)) & \longrightarrow & \text{Ext}^1(M, M) \end{array} \quad (8.29)$$

But

$$H^1(\mathcal{H}om(M, M) \otimes \mathcal{O}_{C_\alpha}) = 0, \quad (8.30)$$

by dimensionality, and thus the map

$$H^1(\mathcal{H}om(M, M) \otimes \omega_X) \rightarrow H^1(\mathcal{H}om(M, M)) \quad (8.31)$$

is surjective. The claim follows immediately. \square

Remark. Note that the pairing between $\text{Ext}^1(M, M \otimes \omega_X)$ and $\text{Ext}^1(M, M)$ restricts to the trivial pairing

$$H^1(\mathcal{H}om(M, M) \otimes \omega_X) \otimes H^1(\mathcal{H}om(M, M)) \rightarrow H^2(\mathcal{H}om(M, M) \otimes \omega_X) = 0. \quad (8.32)$$

Since the overall pairing is perfect, it follows that

$$\dim(\text{im}(\text{Ext}^1(M, M \otimes \omega_X) \rightarrow \text{Ext}^1(M, M))) \geq 2h^1(\mathcal{H}om(M, M)). \quad (8.33)$$

This inequality can be strict, e.g., if the first Fitting scheme of M is 0-dimensional and not contained in C_α . (Indeed, in that case, there is another sheaf with the same invariants and smaller endomorphism sheaf.)

Theorem 8.10. *Let M be a simple, pure 1-dimensional sheaf on the Poisson surface X , transverse to the anticanonical curve. Then M is rigid iff there exists a Poisson birational morphism $\pi : Y \rightarrow X$ and a -2 -curve C on Y disjoint from the anticanonical curve, such that $M \cong \pi_* \mathcal{O}_C(d)$ for some integer d .*

Proof. We first show that sheaves of the form $\pi_* \mathcal{O}_C(d)$ are rigid. Since C is transverse to the exceptional locus, $\mathcal{O}_C(d)$ is the minimal lift of its direct image. Since $\mathcal{O}_C(d)$ is certainly simple, and we can compute

$$\chi_\alpha(\mathcal{O}_C(d), \mathcal{O}_C(d)) = -C^2 = 2, \quad (8.34)$$

we conclude that it is infinitesimally rigid, and thus so is its direct image. Since the image has integral Fitting scheme, the result follows.

Now, suppose that M is rigid, with (integral) support C_0 , and let $\psi : \tilde{C}_0 \rightarrow C_0$ be the normalization of C_0 . Then we observe that

$$\mathcal{H}om(M, M) \subset \psi_* \mathcal{O}_{\tilde{C}_0}, \quad (8.35)$$

with quotient supported on the singular locus of C_0 . Indeed, if M' denotes the quotient of $\psi^* M$ by its torsion subsheaf, then M' is torsion-free, so invertible, on the curve \tilde{C}_0 . This operation is functorial, and an isomorphism on the smooth locus of C_0 , so

$$\mathcal{H}om(M, M) \subset \psi_* \mathcal{H}om(M', M') = \psi_* \mathcal{O}_{\tilde{C}_0}. \quad (8.36)$$

Now, $\dim \text{Hom}(M, M) = h^0(\psi_* \mathcal{O}_{\tilde{C}_0}) = 1$, and thus we find that

$$h^1(\mathcal{H}om(M, M)) \geq h^1(\psi_* \mathcal{O}_{\tilde{C}_0}) = h^1(\mathcal{O}_{\tilde{C}_0}), \quad (8.37)$$

with equality only if $\mathcal{H}om(M, M) = \psi_* \mathcal{O}_{\tilde{C}_0}$. Since M is infinitesimally rigid, the Lemma gives $h^1(\mathcal{H}om(M, M)) = 0$, and thus $\mathcal{H}om(M, M) = \psi_* \mathcal{O}_{\tilde{C}_0}$ and $h^1(\mathcal{O}_{\tilde{C}_0}) = 0$.

In particular, \tilde{C}_0 is a smooth rational curve, and M is the direct image of a torsion-free sheaf on $\text{Spec}(\mathcal{H}om(M, M)) \cong \tilde{C}_0$. Since M is the direct image of an invertible sheaf on a smooth curve,

there exists a Poisson birational morphism $\pi : Y \rightarrow X$ such that π^*M is disjoint from $\pi^*\mathcal{O}_{C_\alpha}$. But then

$$\chi(\pi^*M) = \chi_\alpha(\pi^*M) = \chi_\alpha(M) = 2, \quad (8.38)$$

so that $c_1(\pi^*M)^2 = -2$, and disjointness gives $c_1(\pi^*M) \cdot K_Y = 0$. Since the Fitting scheme of π^*M is the image of \tilde{C}_0 under a birational morphism, it follows that π^*M is an invertible sheaf on a -2 -curve as required. \square

Note that if X is not rational, then the only possible -2 curves on X are components of fibers of the rational ruling of X . Thus rigidity in Hitchin-type systems is essentially only a phenomenon of the rational case; we will explore this further in [15].

References

- [1] A. B. Altman and S. L. Kleiman. Compactifying the Picard scheme. *Adv. in Math.*, 35(1):50–112, 1980.
- [2] I. V. Artamkin. On the deformation of sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52(3):660–665, 672, 1988.
- [3] C. Bartocci and E. Macrì. Classification of Poisson surfaces. *Commun. Contemp. Math.*, 7(1):89–95, 2005.
- [4] F. Bottacin. Poisson structures on moduli spaces of sheaves over Poisson surfaces. *Invent. Math.*, 121(2):421–436, 1995.
- [5] F. Bottacin. Poisson structures on moduli spaces of parabolic bundles on surfaces. *Manuscripta Math.*, 103(1):31–46, 2000.
- [6] P. Deligne. *Cohomologie a support propre et construction du foncteur $f^!$* . Appendix to [7].
- [7] R. Hartshorne. *Residues and duality*. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.
- [8] J. C. Hurtubise and E. Markman. Elliptic Sklyanin integrable systems for arbitrary reductive groups. *Adv. Theor. Math. Phys.*, 6(5):873–978 (2003), 2002.
- [9] J. C. Hurtubise and E. Markman. Surfaces and the Sklyanin bracket. *Comm. Math. Phys.*, 230(3):485–502, 2002.
- [10] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [11] N. M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1990.
- [12] N. M. Katz. *Rigid local systems*, volume 139 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.
- [13] M. Lehn. On the cotangent sheaf of Quot-schemes. *Internat. J. Math.*, 9(4):513–522, 1998.
- [14] A. Okounkov and E. Rains. Noncommutative geometry and Painlevé equations. *Algebra Number Theory*, 9(6):1363–1400, 2015.

- [15] E. M. Rains. Generalized Hitchin systems on rational surfaces. arXiv:1307.4033.
- [16] A. N. Tyurin. Symplectic structures on the moduli spaces of vector bundles on algebraic surfaces with $p_g > 0$. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52(4):813–852, 896, 1988.